# A DYNAMIC MULTILAYER SHALLOW WATER MODEL

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Abstract. We propose a new simple approximation of the viscous primitive equations of the ocean including Coriolis force (1.1), by a multilayer shallow water type model. Using a finite volume type discretization in the vertical direction, we show that our system is a consistent approximation of (1.1). Existence and uniqueness of local in time strong solution is proved for the new model. Finally we design a finite volume numerical scheme, taking advantage of the shallow water type formulation and perform preliminary numerical simulations in 1D to illustrate consistency as well as a dynamic behavior (add or remove layers).

## 1. Introduction and Main Result

The main goal of this article is to propose a simple and numerically efficient model of geophysical flows such as large-scale ocean circulations. Many of these flows are generally described by the incompressible Navier-Stokes equations with a free surface [19]. Due to the mathematical complexity of this system, different approximations are usually performed, which aim in particular at finding a compromise between physical consistency and reasonable computational cost. Going beyond the Boussinesq approximation [23] we start our study by considering an homogeneous fluid (water), with density equal to one. Moreover we use the so-called *hydrostatic approximation*, that is we assume the pressure is hydrostatic and is not an unknown of the problem. Precisely, the departure model consists in the primitive equations of the ocean, given in the conservative form below. We use bold characters to indicate vector valued functions or variables. Hence the 3D velocity of the fluid, for which we separate the horizontal component and the vertical one as $\mathbf{U} = (\mathbf{u}, w)^T \in \mathbb{R}^3$, satisfies in a local frame $(\mathbf{x}, z)$ the set of equations:

$$\left\{ \begin{array}{lcl} \nabla_{\mathbf{x}} \cdot \mathbf{u} + \partial_z w & = & 0\,, \\ \partial_t \mathbf{u} + \nabla_{\mathbf{x}} \cdot (\mathbf{u} \otimes \mathbf{u}) + \partial_z (w\,\mathbf{u}) + \nabla_{\mathbf{x}} p & = & -f\,\mathbf{u}^{\perp} + \mu\,\Delta \mathbf{u}\,, \\ \partial_z p & = & -g\,, \end{array} \right. \tag{1.1}$$

considered for

$$t > 0\,,\ \ (\mathbf{x}, z) \in \Omega_t = \left\{ (\mathbf{x}, z) \in \mathbb{R} \times \mathbb{R}^+ \,|\, z_b(\mathbf{x}) \leqslant z \leqslant \eta(t, \mathbf{x}) \right\}\,,$$

where $z_b$ is the topography (not depending on time) and $\eta$ is the free surface. The fluid depth is given by

$$H(t, \mathbf{x}) = \eta(t, \mathbf{x}) - z_b(\mathbf{x})\,.$$

The constant $\mu > 0$ is the viscosity coefficient and $f > 0$ is the Coriolis parameter also chosen constant. Indeed, in this approximation we consider the latitude on the earth as a constant, and our local frame $(\mathbf{x}, z)$ can be seen as a fixed cartesian frame [23]. Hence the gravitational force is supported by the vertical direction, whose modulus is the gravity constant $g$. The hydrostatic pressure $p$ is therefore given, for all $t$, $\mathbf{x}$, $z$ by:

$$p\,(t, \mathbf{x}, z) \,=\, g\,\left(\eta\,(t, \mathbf{x}) - z\right)\,.$$

The system is completed with boundary conditions. We use the subscript $s$ (*resp.* $b$) to indicate that the function is evaluated at the surface (*resp.* the bottom). On the one hand, it holds a kinematic equation and the continuity of stresses at the free surface:

$$\left\{\begin{array}{lcl} \partial_t \eta + \mathbf{u}_s \cdot \nabla_{\mathbf{x}} \eta & = & w_s \,, \\ \\ \partial_z \mathbf{u}_s & = & \nabla_{\mathbf{x}} \mathbf{u}_s \cdot \nabla_{\mathbf{x}} \eta \,, \end{array}\right. \tag{1.2}$$

when considering the atmospheric pressure equal to zero. At the bottom, we impose no penetration and a Navier type wall law [8], with a constant laminar friction coefficient $\kappa$, that is:

$$\left\{\begin{array}{lcl} \mathbf{u}_b \cdot \nabla_{\mathbf{x}} z_b & = & w_b, \\ \\ \kappa\, \mathbf{u}_b & = & \mu\, \partial_z \mathbf{u}_b. \end{array}\right. \tag{1.3}$$

This set of equations (or more complicated versions), though an approximation of Navier-Stokes, has been widely studied for decades from both theoretical and numerical points of view. See the pionner articles [20] for formal derivations and existence results for wind driven flows; [5] for rigorous justifications. Roughly speaking, this hydrostatic pressure approximation relies on an asymptotic expansion of the Navier-Stokes equations with respect to a small dimensionless parameter $\varepsilon$ (aspect ratio), that is the *shallow water* assumption:

$$\varepsilon = \frac{H_0}{\lambda_0} \ll 1 \tag{1.4}$$

where $H_0$ and $\lambda_0$ are the characteristic depth and the typical horizontal wavelenght of the ocean [19, 23].

Although the primitive equations are simpler than the full Navier-Stokes system, they still contain two main difficulties: non linearity and time dependency of the spatial domain. Therefore many other model are built, either from the Navier-Stokes problem, or from the primitive equations. In particular, one classical way to dispense with the moving spatial domain is to perform an integration of the equations in the vertical direction. This leads to the classical shallow water (*Saint-Venant*) model (see for example the rigorous derivation with flat bottom [17], [13, 15, 21] for a small topography, [6, 7] for an arbitrary one). The main assets of such a model are the reduction of the spatial dimension of the problem and its mathematical properties, leading in particular to a very efficient numerical treatment. Indeed the hyperbolic formulation (away from vacuum) allows the use of robust finite volume schemes, even able to handle dambreak situations and wet/dry front for hydraulic or costal problems [16, 24].
But this system still has also some drawbacks. On the one hand, its good numerical behavior is not fully understood from a mathematical point of view. Indeed, although it is well justified when departing from the Euler equations, its viscous version requires additional assumptions to allow the closure of the system [17, 25, 30]. Moreover, it is not well posed neither in the vacuum nor for large variations of the free surface. On the other hand, considering the solution to this system, one can only reach the mean value of the horizontal velocity in the $z$ direction. Therefore we loose information on the vertical profile of the velocity field.

In the present work, we stay in the context of deep water: we want to propose a consistant approximation of the primitive model (1.1)–(1.3) which reduces the mathematical complexity. Hence, in order to keep information on the vertical profile of the velocity field, while taking advantage of the numerical efficiency of the shallow water formulation, we perform a vertical discretization of the fluid depth $H$, cut into $N$ *thin* layers, and integrate the momentum equation on each layer. Let us emphasize here that the slicing is done in the most simple way, that is:

$$H(t, \mathbf{x}) = \sum_{i=1}^{N} h_i(t, \mathbf{x}) \,,$$

where the intermediate layer heights $h_i$ are all of constant size, say $\overline{h}$, except the lowest and the highest ones, which aims at catching somehow the boundary layers at the bottom and the top of the fluid. It is illustrated in Figure 1 for 4 layers. Hence we define the fluid velocity $\mathbf{u}_i$ in layer $i$ by:

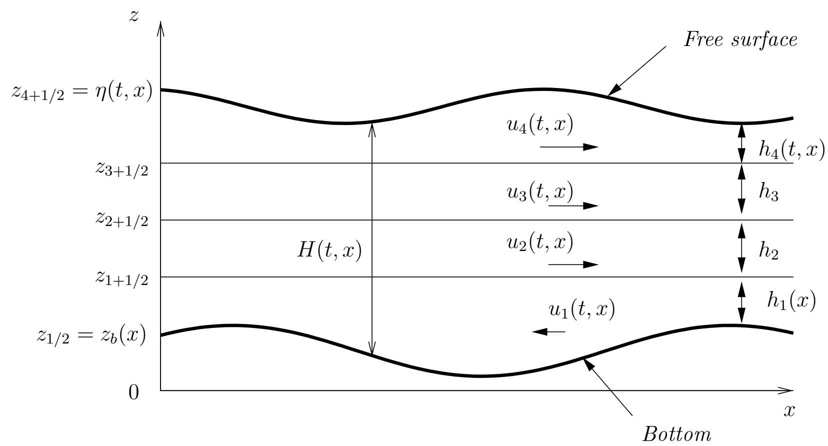


FIGURE 1. *Vertical discretization.*

$$\mathbf{u}_i(t,\mathbf{x}) \;=\; \frac{1}{h_i}\int_{z_{i-1/2}}^{z_{i+1/2}} \mathbf{u}(t,\mathbf{x},z)\,\mathrm{d}z\,,\; 1 \leqslant i \leqslant N\,. \tag{1.5}$$

Then, the $N$-layers model which will be investigated hereafter can be written in 2D as follows. For any $(t,\mathbf{x})$ in $\mathbb{R}^+ \times \mathbb{R}^2$:

$$\left\{\begin{aligned}
&\partial_t H + \nabla_{\mathbf{x}} \cdot \left(\sum_{i=1}^{N} h_i\,\mathbf{u}_i\right) && = 0\,,\\
&\partial_t\left(h_N\,\mathbf{u}_N\right) + \nabla_{\mathbf{x}} \cdot \left(h_n\,\mathbf{u}_N \otimes \mathbf{u}_N + g\,\frac{h_N^2}{2}\right) && = \mu\,\left(\nabla_{\mathbf{x}} \cdot \left(h_N\,\nabla_{\mathbf{x}}\mathbf{u}_N\right) + \mathbf{DU}^z_{N+1/2} - \mathbf{DU}^z_{N-1/2}\right)\\
& && -g\,h_N\,\nabla_{\mathbf{x}} z_b + w_{N-1/2}\,\mathbf{u}_{N-1/2} - w_{N+1/2}\,\mathbf{u}_{N+1/2}\\
& && -f\,\left(h_N\,\mathbf{u}_N\right)^{\perp}\,,\\
&\partial_t\left(h_i\,\mathbf{u}_i\right) + \nabla_{\mathbf{x}} \cdot \left(h_i\,\mathbf{u}_i \otimes \mathbf{u}_i\right) + g\,h_i\,\nabla_{\mathbf{x}} h_N && = \mu\,\left(h_i\,\Delta_{\mathbf{x}}\mathbf{u}_i + \mathbf{DU}^z_{i+1/2} - \mathbf{DU}^z_{i-1/2}\right)\\
& && -g\,h_i\,\nabla_{\mathbf{x}} z_b + w_{i-1/2}\,\mathbf{u}_{i-1/2} - w_{i+1/2}\,\mathbf{u}_{i+1/2}\\
& && -f\,\left(h_i\,\mathbf{u}_i\right)^{\perp}\,,\quad 1 \leqslant i \leqslant N-1\,.
\end{aligned}\right. \tag{1.6}$$

The terms $w_{i+1/2}$, nothing but the values of the vertical velocity at the interfaces $z_{i+1/2}$, provide the mass exchange terms between layers $i$ and $i+1$. They are computed thanks to the integration of the divergence free condition (see Section 2). Precisely, they are defined by:

$$\left\{\begin{aligned}
&w_{1/2} = \mathbf{u}_1 \cdot \nabla_{\mathbf{x}} z_b,\\
&w_{i+1/2} - w_{i-1/2} = -h_i\,\nabla_{\mathbf{x}} \cdot \mathbf{u}_i, \quad 1 \leqslant i \leqslant N-1.
\end{aligned}\right. \tag{1.7}$$

The terms $\mathbf{u}_{i+1/2}$ represent the approximate values of the horizontal velocity at the interfaces $z_{i+1/2}$, given by a centered reconstruction:

$$\mathbf{u}_{i+1/2} = \begin{cases} 0 & \text{if} \quad i = 0\,,\,N\,,\\ \left(h_i\,\mathbf{u}_{i+1} + h_{i+1}\,\mathbf{u}_i\right)/\left(h_{i+1} + h_i\right) & \text{if} \quad 1 \leqslant i \leqslant N-1. \end{cases} \tag{1.8}$$

Finally, the terms $\mathbf{DU}^z_{i+1/2}$ are the $z$-derivatives of the horizontal velocity, evaluated at the interfaces $z_{i+1/2}$ and coming from the vertical viscosity. We choose:

$$\mathbf{DU}^z_{i+1/2} = \begin{cases} \kappa\,\mathbf{u}_1/\mu & \text{if} \quad i=0\,, \\ 2\,(\mathbf{u}_{i+1}-\mathbf{u}_i)/(h_i+h_{i+1}) & \text{if} \quad 1 \leqslant i \leqslant N-1\,, \\ 0 & \text{if} \quad i=N\,. \end{cases} \tag{1.9}$$

The formal derivation of this set of equations in 2D will be obtained in Section 2. Moreover, we will study in this paper the local in time existence of strong solution for the 1D version of the system, that is:

$$\begin{cases} \partial_t H + \partial_x \left( \displaystyle\sum_{i=1}^{N} h_i\,\mathbf{u}_i \right) & = 0\,, \\ \partial_t\,(h_N\,u_N) + \partial_x \left( h_N\,u_N^2 + g\,\dfrac{h_N^2}{2} \right) & = \mu\,\left( \partial_x\,(h_N\,\partial_x u_N) + DU^z_{N+1/2} - DU^z_{N-1/2} \right) \\ & \quad - g\,h_N\,\partial_x z_b + w_{N-1/2}\,u_{N-1/2} - w_{N+1/2}\,u_{N+1/2}\,, \\ \partial_t\,(h_i\,u_i) + \partial_x\,\left(h_i\,u_i^2\right) + g\,h_i\,\partial_x h_N & = \mu\,\left( h_i\,\partial_{xx} u_i + DU^z_{i+1/2} - DU^z_{i-1/2} \right) - g\,h_i\,\partial_x z_b \\ & \quad + w_{i-1/2}\,u_{i-1/2} - w_{i+1/2}\,u_{i+1/2}\,, \quad 1 \leqslant i \leqslant N-1\,. \end{cases} \tag{1.10}$$

where we drop the Coriolis terms which have no meaning in 1D. In order to state the result, we introduce the following notations.
For any function $f$, we note $\|f\|$ ( resp. $\|f\|_k$ ) the $L^2$-norm (resp. $H^k$-norm ) of $f$. If $\mathbf{f}=(f_1,\ldots,f_n)$ is multidimensional, we define its $\mathbf{H}^k$-norm by

$$\|\mathbf{f}\|_k := \sum_{i=1}^{n} \|f_i\|_k\,.$$

Let $B$ be a Banach space, $k$ a non-negative integer and $T$ some positive constant. We denote by $L^k_\infty(0,T;B)$ the Banach space of functions $f$ on $[0,T]$ which have their values in $B$ and are $k$ times differentiable with respect to $t$ and all the derivatives are bounded in $B$. We can now state our main result.

**Theorem 1.1.** *Consider the system (1.10) where $w_{i+1/2}$, $u_{i+1/2}$ and $DU^z_{i+1/2}$ are defined by the 1D versions of (1.7), (1.8) and (1.9), with initial data*

$$(\mathbf{U}, h_N)(0,x) = (\mathbf{U}^0(x), h_N^0(x)) \,\in\, \mathbf{H}^2(\mathbb{R})\,, \tag{1.11}$$

*where $\mathbf{U}=(u_1\,\ldots\,u_N)^T$ is the vector of velocities. Suppose*

$$\inf_{x\in\mathbb{R}} h^0(x) \,\geqslant\, \eta_0 > 0\,,$$

*for some constant $\eta_0$, and note $E = 2\,\|(\mathbf{U}^0, h_N^0)\|_2$. Assume the topography has the regularity $z_b \in \mathcal{C}^2(\mathbb{R})$. Then, there exists a positive constant $T$ such that the Cauchy problem (1.10)-(1.11) has a unique strong solution $(\mathbf{U}, h_N)$ satisfying:*

$$\mathbf{U} \in \mathcal{C}(0,T;\mathbf{H}^2(\mathbb{R})) \cap \mathcal{C}^1(0,T;\mathbf{L}^2(\mathbb{R})) \cap L^2\left(0,T;\mathbf{H}^3(\mathbb{R})\right)\,,$$

$$h_N \in \mathcal{C}(0,T;H^2(\mathbb{R})) \cap \mathcal{C}^1(0,T;H^1(\mathbb{R}))\,.$$

*Moreover, for any $t$ in $[0,T]$,*

$$\forall x \in \mathbb{R}\,,\; h_N(t,x) \geqslant \big(\inf_{x\in\mathbb{R}} h_N^0(x)\big)/2 > 0\,,$$

*and the following energy estimates hold:*

$$\|(\mathbf{U},h_N)(t)\|_2 \leqslant E\,, \quad \left(\int_0^t \|\mathbf{U}(\tau)\|_3^2\,d\tau\right)^{1/2} \leqslant E\,.$$

Theorem 1.1 is stated in 1D for sake of clarity in the computations but the proof, based on energy method of Matsumura and Nishida [22] can be adapted to the two dimensional problem[1], except that we have to choose initial data in $\mathbf{H}^3(\mathbb{R})$, and the solution $(\mathbf{u}_1,\dots,\mathbf{u}_N,h_N)$ get the regularity:

$$\mathbf{u}_i \in \mathcal{C}(0,T;\mathbf{H}^3(\mathbb{R})) \cap \mathcal{C}^1(0,T;\mathbf{H}^1(\mathbb{R})) \cap L^2\left(0,T;\mathbf{H}^4(\mathbb{R})\right)\,, \forall i \in \{1,\dots,N\}\,,$$

$$h_N \in \mathcal{C}(0,T;H^3(\mathbb{R})) \cap \mathcal{C}^1(0,T;H^2(\mathbb{R}))\,.$$

**Remark 1.2.** *This existence result is in good agreement with the ones already existing for classical shallow water systems, in particular the condition on the initial water height, bounded by below by a positive constant. Let us mention, without being exhaustive, for example the works* [10, 18, 26, 27, 29], *treating different kinds of solutions to the Cauchy problem. Unfortunately, concerning weak solutions, no additional energy estimate such as* BD entropy [9] *has been found for (1.10). It is mainly due to the fact that we only have one equation of conservation of mass, in which all the velocities are included.*

**Remark 1.3.** *Of course the existence is only local in time since the model (1.10) blows up when $h_N$ reaches zero at one point. Therefore it gives a criterion to make the model dynamic by removing and adding layers. On the one hand if at a time $t_1$ the highest height $h_N$ becomes too small (say under some non negative threshold), then one removes one layer at the top, and starts again with the model with $N-1$ layers. On the other hand, one can add a layer to the model when the height of the highest layer is large enough. We will see this dynamic behavior in some preliminary numerical simulations of Section 5.*

The rest of the paper is organized as follows. In Section 2 we first derive rigorously the multilayer system (1.6) from the three dimensional free surface primitive model (1.1). Then we briefly compare our model to some existing multilayer models [1, 4] and point out that we do not aim at modelling the same kind of geophysical problems. In Section 3 we prove Theorem 1.1, with the energy method of Matsumura and Nishida [22]. Finally, in Section 4, we design a simple numerical scheme in order to validate our model and perform some numerical experiments in Section 5.

## 2. Derivation of the model and comparison with other multilayer models

2.1. **Derivation.** As it was said in the introduction, we derive our multilayer model from the 3D viscous primitive system with friction and Coriolis terms (1.1)–(1.3) introduced in Section 1. We start by performing the vertical discretization of the water height illustrated in Figure 1:

$$\eta - z_b = H := \sum_{i=1}^{N} h_i, \quad \text{with } h_i = z_{i+1/2} - z_{i-1/2} = O(\overline{h}), \quad 1 \leqslant i \leqslant N, \tag{2.12}$$

where the small constant $\overline{h}$ is fixed and the nodes of discretization are chosen as:

$$\begin{cases} z_{1/2} = z_b(\mathbf{x}), \\ z_{i+1/2} = i\,\overline{h}, \quad 1 \leqslant i \leqslant N-1, \\ z_{N+1/2} = \eta(t,\mathbf{x}). \end{cases} \tag{2.13}$$

Using this vertical discretization and the definition of the velocities (1.5), we claim

[1] The Coriolis term does not add major difficulty since it is a zeroth order term

**Proposition 2.1.** *Assume the variations of the bathymetry are controled as:*

$$\nabla_{\mathbf{x}} z_b \;=\; O\left(\overline{h}\right). \tag{2.14}$$

*Then the multilayer formulation (1.6), where $h_i$, $\mathbf{u}_{i+1/2}$, $w_{i+1/2}$, are given by (2.12), (1.8) and (1.7), is a formal asymptotic approximation in $O\left(\overline{h}^2\right)$ of the primitive equations (1.1)-(1.2)-(1.3).*

*Proof.* On the one hand, the integration through each layer $1 \leqslant i \leqslant N$ of the momentum equation gives:

$$\begin{aligned}\partial_t(h_i\,\mathbf{u}_i) - \Big[\partial_t z\,\mathbf{u}\Big]_{z_{i-1/2}}^{z_{i+1/2}} + \nabla_{\mathbf{x}}\cdot(h_i\,\mathbf{u}_i\otimes\mathbf{u}_i) - \Big[(\nabla_{\mathbf{x}} z\cdot\mathbf{u})\;\mathbf{u}\Big]_{z_{i-1/2}}^{z_{i+1/2}} + \Big[w\,\mathbf{u}\Big]_{z_{i-1/2}}^{z_{i+1/2}} + g\,h_i\,\nabla_{\mathbf{x}}\eta \\ = -f\;(h_i\,\mathbf{u}_i)^{\perp} + \mu\left\{\Big[\partial_z\mathbf{u}\Big]_{z_{i-1/2}}^{z_{i+1/2}} + \nabla_{\mathbf{x}}\cdot\left(\int_{z_{i-1/2}}^{z_{i+1/2}}\nabla_{\mathbf{x}}\mathbf{u}\,\mathrm{d}z\right) - \Big[\nabla_{\mathbf{x}}\mathbf{u}\cdot\nabla_{\mathbf{x}} z\Big]_{z_{i-1/2}}^{z_{i+1/2}}\right\}.\end{aligned} \tag{2.15}$$

Let us notice here that most of the terms between square-brackets will cancel since the inside layer sizes are constant in time and space. On the other hand, by integrating the divergence free condition, we get:

$$w(z_{i+1/2}) - w(z_{i-1/2}) = -\int_{z_{i-1/2}}^{z_{i+1/2}} \nabla_{\mathbf{x}}\cdot\mathbf{u}\,\mathrm{d}z, \quad 1 \leqslant i \leqslant N.$$

It is therefore sufficient to apply Taylor expansions in the vertical direction. Namely, assuming the velocities are smooth enough, we have the following approximations: for all $1 \leqslant i \leqslant N-1$, for all $z \in [z_{i-1/2}, z_{i+1/2}]$:

$$\left\{\begin{array}{lcl} \mathbf{u}(z) & = & \mathbf{u}_i + O\left(\overline{h}\right), \\ \mathbf{u}(z_{i+1/2}) & = & \mathbf{u}_{i+1/2} + O\left(\overline{h}^2\right), \\ \partial_z\mathbf{u}(z_{i+1/2}) & = & 2\,\dfrac{\mathbf{u}_{i+1}-\mathbf{u}_i}{h_i+h_{i+1}} + O\left(\overline{h}^2\right), \\ \displaystyle\int_{z_{i-1/2}}^{z_{i+1/2}} \mathbf{u}\otimes\mathbf{u}\,\mathrm{d}z & = & h_i\,\mathbf{u}_i\otimes\mathbf{u}_i + O\left(\overline{h}^2\right), \\ \displaystyle\int_{z_{i-1/2}}^{z_{i+1/2}} \nabla_{\mathbf{x}}\mathbf{u}\,\mathrm{d}z & = & h_i\,\nabla_{\mathbf{x}}\mathbf{u}_i + O\left(\overline{h}^2\right). \end{array}\right. \tag{2.16}$$

Next, by the use of the boundary conditions at the bottom (1.3) and the order of magnitude of the variations of the bathymetry (2.14), the definition of the approximate reconstructions of the vertical velocity at the interfaces between layers (1.7) yields, for all $0 \leq i \leq N-1$:

$$w_{i+1/2} \;=\; w(z_{i+1/2}) + O\left(\overline{h}^2\right).$$

Let us look at the viscous terms with the previous approximations:

$$\nabla_{\mathbf{x}}\cdot\left(\int_{z_{i-1/2}}^{z_{i+1/2}}\nabla_{\mathbf{x}}\mathbf{u}\,\mathrm{d}z\right) = \left\{\begin{array}{ll} h_1\,\Delta_{\mathbf{x}}\mathbf{u}_1 - \nabla_{\mathbf{x}}\mathbf{u}_b\cdot\nabla_{\mathbf{x}} z_b + O(\overline{h}^2) & \text{if } i=1, \\ h_i\,\Delta_{\mathbf{x}}\mathbf{u}_i + O(\overline{h}^2) & \text{if } i=2,\dots,N-1, \\ \nabla_{\mathbf{x}}\cdot(h_N\,\nabla_{\mathbf{x}}\mathbf{u}_N) + O(\overline{h}^2) & \text{if } i=N. \end{array}\right.$$

Hence, in the equation (2.15) for the lowest layer, the non zero boundary terms of the viscous part cancel each other and we get, using the boundary conditions at the bottom:

$$\begin{aligned}\partial_t\left(h_1\,\mathbf{u}_1\right)+\nabla_{\mathbf{x}}\cdot\left(h_1\,\mathbf{u}_1\otimes\mathbf{u}_1\right)+g\,h_1\,\nabla_{\mathbf{x}}h_N &= -g\,h_1\,\nabla_{\mathbf{x}}z_b-w_{3/2}\,\mathbf{u}_{3/2}-\kappa\,\mathbf{u}_1-f\,\left(h_1\,\mathbf{u}_1\right)^{\perp}\\ &\quad+\mu\,h_1\,\Delta_{\mathbf{x}}\mathbf{u}_1+2\,\mu\,\frac{\mathbf{u}_2-\mathbf{u}_1}{h_1+h_2}+O\left(\overline{h}^2\right).\end{aligned}$$

For the inside layers, we use again the approximations (2.16): the result is obtained easily because the intermediate layer heights are *constant* in time and space. Finally, there is another term in equation (2.15) for the highest layer, coming from the time dependency of the highest layer. It is simplified thanks to the boundary conditions at the free surface (1.2). It leads to:

$$\begin{aligned}\partial_t\left(h_N\,\mathbf{u}_N\right)+\nabla_{\mathbf{x}}\cdot\left(h_N\,\mathbf{u}_N\otimes\mathbf{u}_N\right)+g\,h_N\,\nabla_{\mathbf{x}}h_N &= -g\,h_N\,\nabla_{\mathbf{x}}z_b-w_{N+1/2}\,\mathbf{u}_{N+1/2}-f\,\left(h_N\,\mathbf{u}_N\right)^{\perp}\\ &\quad+\mu\,\nabla_{\mathbf{x}}\cdot\left(h_N\,\nabla_{\mathbf{x}}\mathbf{u}_N\right)-2\,\mu\,\frac{\mathbf{u}_N-\mathbf{u}_{N-1}}{h_N+h_{N-1}}+O\left(\overline{h}^2\right).\end{aligned}$$

To conclude, we drop the $O\left(\overline{h}^2\right)$ and obtain the system (1.6)-(1.7) as a formal approximation of system (1.1)–(1.3) in $O\left(\overline{h}^2\right)$. This ends the proof. □

2.2. **Comparison with other multilayer models.** Let us now briefly compare our model to other multilayer shallow water models, that is the ones introduced by E. Audusse and coauthors [1, 4]. First, we want to point out that if the general framework is somehow similar, the models do not aim at modelling the same phenomena. We focus here on deep water, while the models of [1, 4] mainly treat costal area [2, 3, 4, 12][2]. Actually, we can see our model as an intermediate step for discretization of the primitive model, with an adaptative mesh in the vertical direction. Indeed, we will see in the preliminary numerical results that we can change the number of layers as time goes. Moreover, when we approach a costal zone, our multilayer model could be coupled with a classical shallow water model.
Second, the way of cutting the water height $H$ is different. In [1, 4], the authors "follow" the free surface inside the fluid, as illustrated in Figure 2 for 4 layers. Therefore, this vertical discretization allows to keep

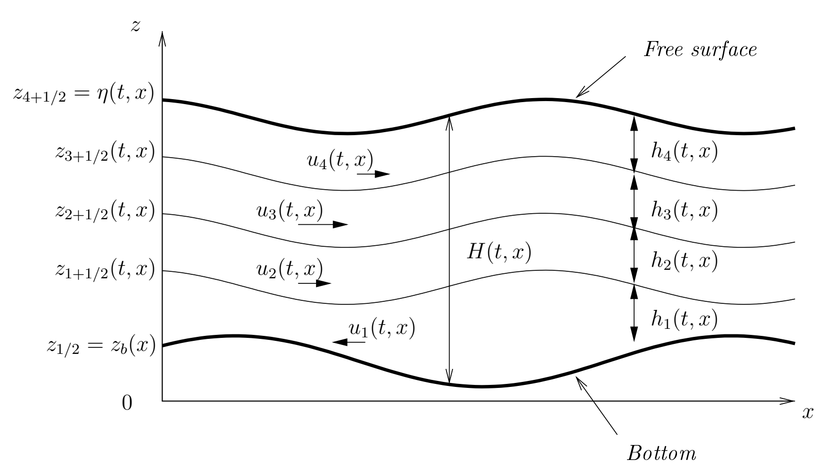


FIGURE 2. *Classical multilayer approach.*

all the good properties of the classical shallow water system: the positivity of the total height immediately gives positivity for all the inside layers and the numerical treatment of the vacuum is also done as for the one layer case. Unfortunately, it keeps also the same mathematical weakness of the classical shallow

[2]Indeed these models are rather derived from the dimensionless Navier-Stokes equations and give a formal approximation in $O(\varepsilon^2)$, where $\varepsilon$ is defined in (1.4).

water model, in particular the closure of the system for the viscous terms[3]. Indeed, in our case we do not need additional assumption on the regimes of viscosity and friction, as in the derivation of the classical viscous shallow water system by J.-F. Gerbeau and B. Perthame [17].

## 3. Well-posedness of the multilayer model

In this section, we study the well-posedness of the 1D multilayer model (1.10) and prove Theorem 1.1. To do so, we rewrite the system under the form of a coupled parabolic-hyperbolic system with source terms. Since the only unknown layer height with our framework is the highest one $h_N$, we will denote it $h$ for sake of clarity. Then, by dividing the equations by the heights, we rewrite the system on the unknown $(\mathbf{U}, h) = (u_1 \, \dots \, u_N, h)^T$ as follows.

$$\begin{cases} \partial_t \mathbf{U} - \mu \, \partial_{xx} \mathbf{U} = \mathbf{S} \,, \\ \\ \partial_t h + \partial_x (h \, u_N) = F \,, \end{cases} \tag{3.17}$$

where the source terms are described below.

$$\begin{cases} \mathbf{S} \; = \; \mathbf{S}_b + \mathbf{S}_l + \mathbf{S}_{nl} \,, \\ \\ F \; = \; w_{N-1/2} \; = \; - \displaystyle\sum_{i=1}^{N-1} \partial_x \left( h_i \, u_i \right) \,, \end{cases}$$

where $\mathbf{S}_b$ refers to the bottom source term

$$\mathbf{S}_b = -g \, \partial_x z_b \, (1, \dots, 1)^T \,,$$

while $\mathbf{S}_l = \left( S_l^1, \dots, S_l^N \right)^T$ and $\mathbf{S}_{nl} = \left( S_{nl}^1, \dots, S_{nl}^N \right)^T$ are respectively the linear and the non linear sources, that is:

$$\begin{cases} S_l^i & = & -g \, \partial_x h \,, \quad 1 \leqslant i \leqslant N \\ \\ S_{nl}^1 & = & 2 \, \mu \, \dfrac{u_2 - u_1}{h_1 \, (h_1 + h_2)} - \dfrac{\kappa}{h_1} \, u_1 \\ \\ & & -\partial_x \left( u_1^2 \right) - \dfrac{1}{h_1} \, \left( u_{3/2} \, w_{3/2} - u_1 \, w_{1/2} \right) \,, \\ \\ S_{nl}^i & = & 2 \, \mu \, \dfrac{u_{i+1} - u_i}{h_i \, \left( h_i + h_{i+1} \right)} - 2 \, \mu \, \dfrac{u_i - u_{i-1}}{h_i \, \left( h_i + h_{i-1} \right)} \\ \\ & & -\partial_x \left( u_i^2 \right) - \dfrac{1}{h_i} \, \left( u_{i+1/2} \, w_{i+1/2} - u_{i-1/2} \, w_{i-1/2} \right) \,, \quad 2 \leqslant i \leqslant N-1 \,, \\ \\ S_{nl}^N & = & -2 \, \mu \, \dfrac{u_N - u_{N-1}}{h \, \left( h + h_{N-1} \right)} + 4 \, \mu \, \dfrac{\partial_x h \, \partial_x u_N}{h} \\ \\ & & -\dfrac{1}{2} \, \partial_x \left( u_N^2 \right) + \dfrac{1}{h} \, \left( u_{N-1/2} - u_N \right) \, w_{N-1/2} \,. \end{cases}$$

Hence, we can sum up by considering that the source term $\mathbf{S}_{nl}$ is roughly composed of three kinds of non linearities, that is

$$u/h \,, \quad u \, \partial_x u/h \,, \quad \partial_x h \, \partial_x u/h \,.$$

[3] The viscous terms in [1, 4] are chosen as the one in the classical shallow water system, but the derivation is justified in the zero viscosity case.

Consequently, in order to simplify the next calculations, we will only consider the simpler hyperbolic-parabolic problem

$$\left\{ \begin{array}{l} \partial_t \mathbf{U} - \mu\, \partial_{xx} \mathbf{U} = \mathbf{S}_b + \mathbf{S}_l + \mathbf{S}_{nl}\,, \\ \\ \partial_t h + \partial_x (h\, u_N) = F\,, \end{array} \right. \tag{3.18}$$

where $F$, $\mathbf{S}_b$, $\mathbf{S}_l$ are not changed, while the nonlinear source is simplified as

$$\mathbf{S}_{nl} = \sum_{k=1}^{3} \mathbf{S}_k\,,$$

where

$$\mathbf{S}_1 = \frac{\mathbf{U}}{h}\,,\;\; \mathbf{S}_2 = \frac{u_N}{h}\,\partial_x \mathbf{U}\,,\;\; \mathbf{S}_3 = \frac{1}{h}\,\partial_x h\,\partial_x \mathbf{U}\,.$$

The proof of Theorem 1.1 is divided into three parts. In the first subsection we perform some estimates on the source terms for the simpler problem (3.18). Next we solve a linearized version of system (3.18) and derive energy estimates. Finally, we build a recursive sequence of solutions of linear systems, and show a convergence to the strong solution we are looking for.

3.1. **Estimates on the source terms.** We first recall a classical lemma of analysis which will be useful to estimate the source terms [28].

**Lemma 3.1** (Moser estimate)**.** *Let $k \in \mathbb{N}$ and $n \in \mathbb{N}^*$. Suppose $f, g \in H^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then $f, g \in H^k(\mathbb{R}^n)$ and there exists a positive constant $C$ such that*

$$\|fg\|_k \leqslant C\, \left(\|f\|_k\, \|g\|_\infty + \|g\|_k\, \|f\|_\infty\right)\,.$$

We are now ready to state the estimations of the source terms.

**Lemma 3.2.** *Let $\mathbf{U}(t,\cdot)$, $h(t,\cdot) \in \mathbf{H}^2(\mathbb{R})$ such as, $h(t,x) \geq \eta_0 > 0$, for some constant $\eta_0$. Then it holds:*

- *$(\mathbf{S}, F) \in \mathbf{H}^1(\mathbb{R})$ and we have the following estimates.*

$$\|\mathbf{S}\|_1 \;\leqslant\; C(\eta_0)\, \|(\mathbf{U},h)\|_2 \left(1 + \|(\mathbf{U},h)\|_2\right), \tag{3.19}$$

$$\|F\|_1 \;\leqslant\; C_b\, \|\mathbf{U}\|_2\,, \tag{3.20}$$

*where $C(\eta_0)$, $C_b$ are positive constants depending respectively, only on $\eta_0$ and the topograhy $z_b$.*

- *If moreover $\mathbf{U} \in \mathbf{H}^3(\mathbb{R})$, then $F \in H^2(\mathbb{R})$ and there exists some constant $C_b$ such that:*

$$\|F\|_2 \leqslant C_b\, \|\mathbf{U}\|_3\,.$$

- *Let $(\mathbf{U},h)\,,(\mathbf{U}',h') \in \mathbf{H}^2(\mathbb{R})$ such that, $\forall\,(t,x) \in \mathbb{R}^+ \times \mathbb{R}$*

$$\|(\mathbf{U},h)\|_2\,,\, \|(\mathbf{U}',h')\|_2 \leqslant E\,,\quad h\,,\, h' \geq \eta_0 > 0\,, \tag{3.21}$$

*for some constants $E$ and $\eta_0$. Then it holds*

$$\|\mathbf{S}(\mathbf{U},h) - \mathbf{S}(\mathbf{U}',h')\|_1 \;\leqslant\; C(\eta_0)\, \left(1 + E + E^2\right)\, \|(\mathbf{U}-\mathbf{U}', h-h')\|_2\,, \tag{3.22}$$

$$\|F(\mathbf{U}) - F(\mathbf{U}')\|_1 \;\leqslant\; C_b\, \|\mathbf{U}-\mathbf{U}'\|_2\,, \tag{3.23}$$

*where $C(\eta_0)$, $C_b$ are positive constants independent of $E$.*

*Proof.* The estimate on $F$ is directly obtained from the definition

$$F = \partial_x z_b\, u_1 - \sum_{i=1}^{N-1} h_i\, \partial_x u_i\,.$$

We have, for $k = 1$ or 2:

$$\|F\|_k \leqslant C_b\, \|\mathbf{U}\|_{k+1}\,.$$

Next the linear sources are estimated as

$$\begin{cases} \|\mathbf{S}_b\|_1 \leqslant C\,\|z_b\|_2\,, \\ \\ \|\mathbf{S}_l\|_1 \leqslant C\,\|h\|_2\,. \end{cases}$$

Then we estimate the non linear source part $\mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3$. It follows from Lemma 3.1 and the classical Sobolev embedding

$$H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})\,,$$

that we have the estimates:

$$\|\mathbf{S}_1\|_1 \leqslant \frac{C}{\eta_0}\,\|\mathbf{U}\|_1\,,\ \ \|\mathbf{S}_2\|_1 \leqslant \frac{C}{\eta_0}\,\|\mathbf{U}\|_2^2\,,\ \ \|\mathbf{S}_3\|_1 \leqslant \frac{C}{\eta_0}\,\|h\|_2\,\|\mathbf{U}\|_2\,.$$

It gives immediately (3.19). Now we note $\mathbf{S}=\mathbf{S}(\mathbf{U},h)$ and $\mathbf{S}'=\mathbf{S}(\mathbf{U}',h')$. We compute the difference

$$\begin{cases} \mathbf{S}_l^i-\mathbf{S}_l'^i &= -g\,\partial_x(h-h')\,\forall\, i\,, \\ \\ \mathbf{S}_1-\mathbf{S}_1' &= \dfrac{1}{h}\,\big(\mathbf{U}-\mathbf{U}'\big)+\dfrac{h'-h}{h\,h'}\,\mathbf{U}'\,, \\ \\ \mathbf{S}_2-\mathbf{S}_2' &= \dfrac{u_N}{h}\,\partial_x(\mathbf{U}-\mathbf{U}')+\dfrac{u_N-u_N'}{h}\,\partial_x\mathbf{U}'+\dfrac{u_N'}{h\,h'}\,(h'-h)\,\partial_x\mathbf{U}'\,, \\ \\ \mathbf{S}_3-\mathbf{S}_3' &= \dfrac{\partial_x h}{h}\,\partial_x(\mathbf{U}-\mathbf{U}')+\dfrac{u_N-u_N'}{h}\,\partial_x\mathbf{U}'+\dfrac{u_N'}{h\,h'}\,(h'-h)\partial_x\mathbf{U}'\,. \end{cases}$$

Then, applying Lemma 3.1 and using (3.21), we obtain

$$\begin{cases} \|\mathbf{S}_l-\mathbf{S}_l'\|_1 &\leqslant C\,\|h-h'\|_2\,, \\ \\ \|\mathbf{S}_1-\mathbf{S}_1'\|_1 &\leqslant \dfrac{C}{\eta_0}\,\|\mathbf{U}-\mathbf{U}'\|_1+\dfrac{C}{\eta_0^2}\,E\,\|h'-h\|_1\,, \\ \\ \|\mathbf{S}_2-\mathbf{S}_2'\|_1 &\leqslant \dfrac{C}{\eta_0}\,E\,\|\mathbf{U}-\mathbf{U}'\|_2+\dfrac{C}{\eta_0^2}\,E^2\,\|h-h'\|_1\,, \\ \\ \|\mathbf{S}_3-\mathbf{S}_3'\|_1 &\leqslant \dfrac{C}{\eta_0}\,E\,\|\mathbf{U}-\mathbf{U}'\|_2+\dfrac{C}{\eta_0}\,E\,\|h-h'\|_2+\dfrac{C}{\eta_0^2}\,E^2\,\|h'-h\|_1\,. \end{cases}$$

Adding these inequalities, we get (3.22). Finally, the inequality (3.23) is straight forward since $F$ is linear with respect to $\partial_x\mathbf{U}$. □

Next we give estimate of the commutator of the transport operator $\partial_t+u_N\,\partial_x$ and the second order space differential operator $\partial_{xx}$.

**Lemma 3.3.** *We assume $u_N\in H^2(\mathbb{R})$ with*

$$\|u_N\|_2 \leqslant E$$

*for some positive constant $E$, and define the differential operator*

$$\mathcal{L}_{u_N} := \partial_t+u_N\,\partial_x\,.$$

*Then, for any $h\in L^0_\infty(0,T;H^2(\mathbb{R}))$, we have:*

$$\Big\|\partial_{xx}\Big(\mathcal{L}_{u_N}\,(h)\Big)\;-\;\mathcal{L}_{u_N}\big(\partial_{xx}h\big)\Big\| \leqslant C\,E\,\|h\|_2\,,$$

*Proof.* We only compute:

$$\partial_{xx}\Big(\mathcal{L}_{u_N}\,(h)\Big)\;-\;\mathcal{L}_{u_N}\,\big(\partial_{xx}h\big)\;=\;2\,\partial_x u_N\,\partial_{xx}h+\partial_{xx}u_N\,\partial_x h\,.$$

Hence, using Lemma 3.1 yields:

$$\Big\|\partial_{xx}\Big(\mathcal{L}_{u_N}\,(h)\Big)\;-\;\mathcal{L}_{u_N}\,\big(\partial_{xx}h\big)\Big\|\;\leqslant\;2\,E\,\|\partial_{xx}h\|+E\,\|\partial_x h\|_1\,.$$

□

In the next subsection, we obtain energy estimates and study a linearized version of the multilayer system.

3.2. **Study of the linearized problem.** Let us introduce a linearized version of system (3.18):

$$\left\{\begin{aligned}&\partial_t\mathbf{U}-\mu\,\partial_{xx}\mathbf{U}\;=\;\mathbf{S}(\tilde{\mathbf{U}},\tilde{h},\partial_x\tilde{\mathbf{U}},\partial_x\tilde{h}):=\tilde{\mathbf{S}},\\&\mathcal{L}_{\tilde{u}_N}\;(h)\;=\;F-\tilde{h}\,\partial_x u_N:=f.\end{aligned}\right.\tag{3.24}$$

In order to study the well-posedness of this coupled linear parabolic-hyperbolic problem, we first solve the parabolic system, and next the transport equation on $h$ by considering the right hand side

$$f=-\tilde{h}\,\partial_x u_N-\sum_{i=1}^{N-1}\partial_x\left(h_i\,u_i\right)$$

as a known function. Thus, we will first study separately the following Cauchy problems, one parabolic system

$$\left\{\begin{aligned}&\left(\partial_t-\mu\,\partial_{xx}\right)\big(\mathbf{U}\big)\;=\;\tilde{\mathbf{S}}\,,\\&\mathbf{U}(0,x)=\mathbf{U}^0\;\in\;\mathbf{H}^2(\mathbb{R})\,,\end{aligned}\right.\tag{A}$$

and one hyperbolic scalar equation:

$$\left\{\begin{aligned}&\mathcal{L}_{\tilde{u}_N}\,\big(h\big)\;=\;f\,,\\&h(0,x)=h^0\;\in\;H^2(\mathbb{R})\,.\end{aligned}\right.\tag{B}$$

**Proposition 3.4.** *Let* $\tilde{\mathbf{S}}\in\mathcal{C}(0,T;\mathbf{H}^1(\mathbb{R}))$ *for some* $T>0$. *Then the initial value problem (A) has a unique strong solution* $\mathbf{U}$ *which satisfies:*

$$\mathbf{U}\;\in\;\mathcal{C}(0,T;\mathbf{H}^2(\mathbb{R}))\cap\mathcal{C}^1(0,T;\mathbf{L}^2(\mathbb{R}))\cap L^2(0,T;\mathbf{H}^3(\mathbb{R}))\,.$$

*Moreover, there exist two positive constants* $C_1$ *and* $C_2$*, depending only on the viscosity, such that for any* $t$ *in* $[0,T]$*:*

$$\|\mathbf{U}(t)\|_2^2+C_1\int_0^t\|\mathbf{U}(\tau)\|_3^2\,d\tau\;\leqslant\;e^t\left(\|\mathbf{U}(0)\|_2^2+C_2\int_0^t\|\tilde{\mathbf{S}}(\tau)\|_1^2\,d\tau\right).\tag{3.25}$$

*Proof.* First, the energy inequality is obtained in a classical way. Multiplying the system by $\mathbf{U}$ and integrating in space, one gets, for any $\alpha>0$:

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\mathbf{U}\|^2+\mu\,\|\partial_x\mathbf{U}\|^2\quad\leqslant\quad\frac{\alpha}{2}\,\|\mathbf{U}\|^2+\frac{1}{2\,\alpha}\,\|\tilde{\mathbf{S}}\|^2\,.$$

Next, we differentiate with respect to $x$, multiply by $\partial_x\mathbf{U}$ and integrate in space, it gives, for any $\alpha>0$:

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\partial_x\mathbf{U}\|^2+\mu\,\|\partial_{xx}\mathbf{U}\|^2\quad\leqslant\quad\frac{\alpha}{2}\,\|\partial_x\mathbf{U}\|^2+\frac{1}{2\,\alpha}\,\|\partial_x\tilde{\mathbf{S}}\|^2\,.$$

Finally, we compute the second order space derivative, multiply by $\partial_{xx}\mathbf{U}$ and integrate in space. Here, since we have too many derivatives on the source term $\tilde{\mathbf{S}}$, we integrate by parts the right hand side as follows: for any $\alpha > 0$,

$$\left|\int_{\mathbb{R}} \partial_{xx}\tilde{\mathbf{S}}\,\partial_{xx}\mathbf{U}\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}} \partial_{x}\tilde{\mathbf{S}}\,\partial_{xxx}\mathbf{U}\,\mathrm{d}x\right| \leqslant \frac{\alpha}{2}\,\|\partial_{xxx}\mathbf{U}\|^2 + \frac{1}{2\,\alpha}\,\|\partial_x\tilde{\mathbf{S}}\|^2 .$$

Now we choose $\alpha$ such that $C_1 := 2\,\mu - \alpha > 0$, we add the previous inequalities and get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|\mathbf{U}\|_2^2 + C_1\,\|\mathbf{U}\|_3^2 \quad \leqslant \quad \alpha\,\|\mathbf{U}\|_2^2 + \frac{1}{\alpha}\,\|\tilde{\mathbf{S}}\|_1^2 .$$

We end the proof by applying the Gronwall Lemma.
This *a priori* estimate gives uniqueness of the solution. Concerning the proof of existence of solution, we introduce $K^t$ the Green kernel of the operator $\partial_t - \mu\,\partial_{xx}$. Then, Duhamel's formula gives a solution $\mathbf{U} = (u_1, \ldots, u_N)^T$ of problem (A) defined by:

$$\forall\,(t,x)\,\in\,[0,T[\times\mathbb{R}\,,\; u_i(t,x) = K^t * u_i^0 + \int_0^t K^{t-s} * \tilde{\mathbf{S}}^i(s)\,\mathrm{d}s\,,\; i = 1\,,\ldots,N\,.$$

We deduce immediately the smoothness of $\mathbf{U}$: it lies in $\mathcal{C}(0,T;\mathbf{H}^2(\mathbb{R}))$. To get more regularity in time, we observe that:

$$\partial_t\mathbf{U} = \mu\,\partial_{xx}\mathbf{U} + \tilde{\mathbf{S}}\,\in\,\mathcal{C}(0,T;\mathbf{L}^2(\mathbb{R}))\,.$$

Hence $\mathbf{U}\,\in\,\mathcal{C}(0,T;\mathbf{H}^2(\mathbb{R}))\cap\mathcal{C}^1(0,T;\mathbf{L}^2(\mathbb{R}))$. □

We can now solve the Cauchy problem (B), considering the right hand side $f$ as a known function.

**Proposition 3.5.** *Let $\tilde{u}_N \in \mathcal{C}(0,T;H^2(\mathbb{R}))$ and $f \in \mathcal{C}(0,T;H^k(\mathbb{R}))$ for $k = 1$ or 2, and $T > 0$. Denote*

$$E := \sup_{0\leqslant t\leqslant T}\,\{\|\tilde{u}_N(t)\|_2\}\;.$$

*Then the initial value problem (B) has a unique strong solution $h$ which satisfies:*

$$h\,\in\,\mathcal{C}(0,T;H^k(\mathbb{R}))\cap\mathcal{C}^1(0,T;H^{k-1}(\mathbb{R}))\,.$$

*Moreover, there exists a positive constant $C_3$, depending only on the dimension of the space, such that, for all $t$ in $[0,T]$:*

$$\|h(t)\|_k \quad\leqslant\quad e^{C_3\,E\,t}\left(\|h(0)\|_k + \int_0^t e^{-C_3\,E\,\tau}\,\|f(\tau)\|_k\,d\tau\right)\,,\quad k = 1 \text{ or } 2\,. \tag{3.26}$$

*Proof.* As in Proposition 3.4, we first obtain the energy estimate. Multiplying the equation by $h$ and integrating in space, we get

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|h\|^2 \quad=\quad -\int_{\mathbb{R}} \tilde{u}_N\,\partial_x\left(\frac{h^2}{2}\right)\,\mathrm{d}x + \int_{\mathbb{R}} f\,h\,\mathrm{d}x\,.$$

We apply an integration by parts on the first term of the right hand side, and estimate the second term with Hölder inequality. It yields

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|h\|^2 \quad\leqslant\quad \frac{1}{2}\,E\,\|h\|^2 + \|f\|\,\|h\|\,.$$

Remove the square on the $L^2$-norms and get, for some constant $C > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|h\| \,\leqslant\, C\,E\,\|h\| + \|f\|\,. \tag{3.27}$$

Next, we differentiate the equation and multiply by $\partial_x h$. We note that

$$\int_{\mathbb{R}} \partial_x\big(\tilde{u}_N\,\partial_x h\big)\,\partial_x h\,\mathrm{d}x = \frac{1}{2}\,\int_{\mathbb{R}} \partial_x\tilde{u}_N\,\big(\partial_x h\big)^2\,\mathrm{d}x\,,$$

and get, as previously, the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|\partial_x h\| \,\leqslant\, C\,E\,\|\partial_x h\| + \|\partial_x f\|\,. \tag{3.28}$$

Adding (3.27) and (3.28), it gives (3.26) for $k=1$ thanks to the Gronwall Lemma. For $k=2$, we need the estimate of commutator between the transport operator and $\partial_{xx}$, already proved in Lemma 3.3:

$$\left\|\partial_{xx}\Big(\mathcal{L}_{\tilde{u}_N}\left(h\right)\Big) \;-\; \mathcal{L}_{\tilde{u}_N}\left(\partial_{xx}h\right)\right\| \,\leqslant\, C\,E\,\|h\|_2\,.$$

Hence, by differentiating again the equation, multiplying by $\partial_{xx}h$ and using the previous estimate, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|\partial_{xx} h\| \,\leqslant\, C\,E\,\|\partial_{xx} h\| + \|\partial_{xx} f\|\,. \tag{3.29}$$

Finally, adding (3.29) with (3.27) and (3.28) and applying the Gronwall Lemma, we obtain (3.26) for $k=2$.

Next, we study the existence of solution for problem (B). We define the characteristic curve $X$ associated to the equation:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}X = \tilde{u}_N(t,X)\,, \\[2ex] X(t=t_0) = x_0\,. \end{cases}$$

Then, solution of (B) reads:

$$h(t,X(t,x)) = h(0,X(0,x)) + \int_0^t f(s,X(s,x))\,\mathrm{d}s \quad \forall t\in[0,T]\,.$$

We thus deduce that $h\,\in\,\mathcal{C}(0,T;H^1(\mathbb{R}))\cap\mathcal{C}^1(0,T;L^2(\mathbb{R}))$. For $k=2$, we differentiate (B) with respect to $x$. Then $\phi:=\partial_x h$ is solution of

$$\begin{cases} \partial_t\phi \,+\, \tilde{u}_N\,\partial_x\phi = \partial_x f \,-\, \partial_x\tilde{u}_N\,\phi\,, \\[2ex] \phi(0,x) = \partial_x h^0 \,\in\, H^1(\mathbb{R})\,. \end{cases}$$

We solve this initial value problem by the iteration:

$$\phi^{(0)}(t,x) = \partial_x h^0(x)\,,\,\forall\,(t,x)\,\in\,[0,T]\times\mathbb{R}\,,$$

and $\phi^{(j)}$, for $j\geq 1$ is the solution of

$$\begin{cases} \partial_t\phi^{(j)} \,+\, \tilde{u}_N\,\partial_x\phi^{(j)} &=\quad \partial_x f \,-\, \partial_x\tilde{u}_N\,\phi^{(j-1)}\,, \\[2ex] \phi^{(j)}(0,x) &=\quad \partial_x h^0(x) \quad \forall\,x\,\in\,\mathbb{R}\,. \end{cases}$$

Since

$$\|\partial_x f \,-\, \partial_x\tilde{u}_N\,\phi^{(j-1)}\|_1 \,\leqslant\, \|f\|_2 + C\,E\,\|\phi^{(j-1)}\|_1\,,$$

the approximation $\phi^{(j)}$ lies in $\mathcal{C}(0,T;H^1(\mathbb{R}))$. To get the convergence of $\left(\phi^{(j)}\right)_j$ to $\partial_x h$, we observe that

$$\mathcal{L}_{\tilde{u}_N}\,\left(\phi^{(j+1)}-\phi^{(j)}\right) = \partial_x\tilde{u}_N\,\left(\phi^{(j)}-\phi^{(j-1)}\right)\,,$$

and apply $j$ times the energy estimate (3.26) to get

$$\begin{aligned} \left\|\phi^{(j+1)}-\phi^{(j)}\right\|_1 \,&\leqslant\, e^{C_3\,E\,t}\int_0^t e^{-C_3\,E\,\tau}\,C_3\,E\|\phi^{(j)}-\phi^{(j-1)}\|_1\,\mathrm{d}\tau \\ &\leqslant\cdots\leqslant\, e^{C_3\,E\,t}\,\frac{(C_3\,E\,t)^j}{j!}\,\left(2\,\|\partial_x h^0\|_1 + \int_0^t e^{-C_3\,E\,\tau}\,\|\partial_x f(\tau)\|_1\,\mathrm{d}\tau\right)\,, \end{aligned}$$

which tends to zero as $j$ goes to $+\infty$. This gives the convergence of $(\phi^{(j)})_j$ to $\partial_x h$ in $H^1$, and then the $H^2$-regularity of $h$. ☐

Combining the previous propositions, we obtain existence for the full linearized problem (3.24).

**Proposition 3.6.** *Let* $\tilde{\mathbf{S}} \in \mathcal{C}(0,T;\mathbf{H}^1(\mathbb{R}))$ *and* $\tilde{u}_N\,,\tilde{h}\,\in \mathcal{C}(0,T;H^2(\mathbb{R}))$ *for some* $T>0$*. Then the initial value problem*

$$\begin{cases} \partial_t \mathbf{U} - \mu\,\partial_{xx}\mathbf{U} = \tilde{\mathbf{S}}\,, \\ \\ \partial_t h + \tilde{u}_N\,\partial_x h = F - \tilde{h}\,\partial_x u_N\,, \end{cases}$$

*has a unique strong solution* $(\mathbf{U},h)$ *which satisfies:*

$$\mathbf{U}\,\in\,\mathcal{C}(0,T;\mathbf{H}^2(\mathbb{R}))\cap\mathcal{C}^1(0,T;\mathbf{L}^2(\mathbb{R}))\cap L^2(0,T;\mathbf{H}^3(\mathbb{R}))\,,$$

$$h\,\in\,\mathcal{C}(0,T;H^2(\mathbb{R}))\cap\mathcal{C}^1(0,T;H^1(\mathbb{R}))\,.$$

*Moreover, there exist two positive constants* $K$ *and* $C$*, only depending on the topography and the viscosity coefficient, such that, for all* $t$ *in* $[0,T]$*:*

$$\|(\mathbf{U},h)(t)\|_2\,,\ \left(\int_0^t \|\mathbf{U}(\tau)\|_3^2\,d\tau\right)^{1/2} \leqslant K\,e^{C\,(1+E)^2\,t}\left\{\|(\mathbf{U}^0,h^0)\|_2 + \left(\int_0^t \|\tilde{\mathbf{S}}(\tau)\|_1^2\,d\tau\right)^{1/2}\right\}\,, \tag{3.30}$$

*where* $E := max\left\{\underset{0\leqslant t\leqslant T}{sup}\,\{\|\tilde{u}_N(t)\|_2\}\,,\ \underset{0\leqslant t\leqslant T}{sup}\,\left\{\|\tilde{h}(t)\|_2\right\}\right\}$.

*Proof.* We first obtain the energy estimate (3.30). On the one hand, we observe that the right hand side of the transport equation verifies

$$f = -\tilde{h}\,\partial_x u_N - \sum_{k=1}^{N-1}\partial_x\,(h_i\,u_i)\,\in\,\mathcal{C}(0,T;H^1(\mathbb{R}))\cap L^2\left(0,T;H^2(\mathbb{R})\right)\,,$$

and we have the estimate

$$\|f\|_2\,\leqslant\,C_b\,(1+E)\,\|\mathbf{U}\|_3\,.$$

Therefore, applying Cauchy-Schwarz inequality:

$$\int_0^t e^{-C_3\,E\,\tau}\,\|f(\tau)\|_k\,\mathrm{d}\tau\,\leqslant\,e^{C_4\,(1+E)^2\,t}\left(\int_0^t \|\mathbf{U}(\tau)\|_3^2\,\mathrm{d}\tau\right)^{1/2}\,, \tag{3.31}$$

for some constant $C_4$ depending on $C_b$, $C_3$. On the other hand, from the inequality (3.25) we deduce that there exist two constant depending only on the viscosity $C_1'$, $C_2'$ such that

$$\|\mathbf{U}(t)\|_2,\ \left(\int_0^t \|\mathbf{U}(\tau)\|_3^2\,\mathrm{d}\tau\right)^{1/2}\ \leq\,C_1'\,e^{C_2'\,t}\left[\|\mathbf{U}^0\|_2 + \left(\int_0^t \|\tilde{\mathbf{S}}(\tau)\|_1^2\,\mathrm{d}\tau\right)^{1/2}\right]. \tag{3.32}$$

Injecting this estimate in (3.31) yields

$$\int_0^t e^{-C_3\,E\,\tau}\,\|f(\tau)\|_k\,\mathrm{d}\tau\,\leqslant\,C_1'\,e^{C_5\,(1+E)^2\,t}\left[\|\mathbf{U}^0\|_2 + \left(\int_0^t \|\tilde{\mathbf{S}}(\tau)\|_1^2\,\mathrm{d}\tau\right)^{1/2}\right]\,, \tag{3.33}$$

where $C_5 = C_2' + C_4$. Finally, we add (3.26) and (3.32), and control the exponentials to obtain (3.30).

This gives the uniqueness for the solution. Let us now prove the existence of solution. On the one hand, the existence of $\mathbf{U}$ follows from Proposition 3.4, and we have the regularity

$$\mathbf{U}\,\in\,\mathcal{C}(0,T;\mathbf{H}^2(\mathbb{R}))\cap\mathcal{C}^1(0,T;\mathbf{L}^2(\mathbb{R}))\cap L^2(0,T;\mathbf{H}^3(\mathbb{R}))\,,$$

which gives

$$f\,\in\,\mathcal{C}(0,T;\mathbf{H}^1(\mathbb{R}))\cap\mathcal{C}^1(0,T;\mathbf{L}^2(\mathbb{R}))\cap L^2(0,T;\mathbf{H}^2(\mathbb{R}))\,.$$

So we can apply Proposition 3.5 for $k=1$ to obtain the existence of $h$ in $\mathcal{C}(0,T;\mathbf{H}^1(\mathbb{R}))$. To get more regularity in space, we differentiate the problem with respect to $x$:

$$\begin{cases} \partial_t\left(\partial_x \mathbf{U}\right) - \mu\,\partial_{xx}\left(\partial_x \mathbf{U}\right) = \partial_x \tilde{\mathbf{S}}\,, \\ \partial_t\left(\partial_x h\right) + \tilde{u}_N\,\partial_x\left(\partial_x h\right) = \partial_x f - \partial_x \tilde{u}_N\,\partial_x h\,, \\ \left(\partial_x \mathbf{U}^0,\,\partial_x h^0\right)\,\in\,\mathbf{H}^1(\mathbb{R})\,. \end{cases}$$

Noticing that

$$\left\|\partial_x f\right\|_1 \,\leqslant\, C\,E\,\|\partial_x \mathbf{U}\|_1\,,$$

we can solve this problem by the same iteration process as in the lattest part of Proposition 3.5, this concludes the proof. $\square$

In order to obtain the solution to the nonlinear initial value problem (3.18), we will build a convergent sequence, this is the last part of the proof of Theorem 1.1.

3.3. **Iterative scheme.** We construct a recursive sequence $(\mathbf{U}^{(j)},h^{(j)}) = (u_1^{(j)},\ldots,u_N^{(j)},h^{(j)})_{j\in\mathbb{N}}$ as follows.

$$\forall\,(t,x)\,\in\,[0,T]\times\mathbb{R}\,,\;(\mathbf{U}^{(0)},h^{(0)})(t,x) = (\mathbf{U}^0,h^0)(x)\in\mathbf{H}^2(\mathbb{R})\,,$$

and for all $j\in\mathbb{N}$, $(\mathbf{U}^{(j+1)},h^{(j+1)})$ solves the initial value problem:

$$(\mathcal{P}_{j+1})\qquad \begin{cases} \left(\partial_t - \mu\,\partial_{xx}\right)\left(\mathbf{U}^{(j+1)}\right) = \mathbf{S}^{(j)}\,, \\ \mathcal{L}_{u_N^{(j)}}\left(h^{(j+1)}\right) = F^{(j,j+1)}\,, \\ (\mathbf{U}^{(j+1)},h^{(j+1)})(t=0) = (\mathbf{U}^0,h^0)\,, \end{cases}$$

where the sequence of source terms is given by, for any $j\in\mathbb{N}$:

$$\begin{cases} \mathbf{S}^{(j)} = \mathbf{S}\big(\mathbf{U}^{(j)},h^{(j)},\partial_x\mathbf{U}^{(j)},\partial_x h^{(j)}\big)\,, \\ F^{(j,j+1)} = -\displaystyle\sum_{i=1}^{N-1}\partial_x\left(h_i\,u_i^{(j+1)}\right) - \partial_x u_N^{(j+1)}\,h^{(j)}\,. \end{cases}$$

We define the constants

$$\begin{cases} E = 2\,\|(\mathbf{U}^0,h^0)\|_2\,, \\ \eta_0 = \dfrac{1}{2}\displaystyle\inf_{x\in\mathbb{R}} h^0(x)\,. \end{cases}$$

The following lemma gives the existence of the whole sequence.

**Lemma 3.7.** *For suitably small $T>0$, the sequence $(\mathbf{U}^{(j)},h^{(j)})_{j\in\mathbb{N}}$ is well defined and satisfies, for any $t\in[0,T]$ and any $j\in\mathbb{N}$:*

$$\begin{aligned} &\mathbf{U}^{(j)}\in\mathcal{C}(0,T;\mathbf{H}^2(\mathbb{R}))\cap\mathcal{C}^1(0,T;\mathbf{L}^2(\mathbb{R}))\cap L^2(0,T;\mathbf{H}^3(\mathbb{R}))\,, \\ &h^{(j)}\in\mathcal{C}(0,T;H^2(\mathbb{R}))\cap\mathcal{C}^1(0,T;H^1(\mathbb{R}))\,. \end{aligned}\tag{3.34}$$

*Moreover, for all $(t,x)$ in $[0,T]\times\mathbb{R}$ and all $j\in\mathbb{N}$, we have:*

$$\|(\mathbf{U}^{(j)},h^{(j)})(t)\|_2\,,\;\left(\int_0^t\|\mathbf{U}^{(j)}(\tau)\|_3^2\,d\tau\right)^{1/2}\,\leqslant\,E\,,\tag{3.35}$$

$$h^{(j)}(t,x)\,\geq\,\eta_0\,>\,0\,.\tag{3.36}$$

*Proof.* First we initialize the recursion. $(\mathbf{U}^{(0)}, h^{(0)})$ verifies the good conditions by definition. Applying Proposition 3.6, we obtain existence of $(\mathbf{U}^{(1)}, h^{(1)})$ in $\mathcal{C}(0,t;\mathbf{H}^2(\mathbb{R}))$ for any $t>0$. Moreover, applying the characteristic formula to $h^{(1)}$, we get, for any $t>0$:

$$\begin{aligned} h^{(1)}(t,y) &= h^0(X(0,t,y)) + \int_0^t F^{(0,1)}(s,X(s,t,y))\,\mathrm{d}s \\ &\geq 2\,\eta_0 + C(E)\,t \\ &\geq \eta_0 \quad \text{if } t \leqslant T_1 \text{ small enough}, \end{aligned}$$

where $T_1 = T_1(\eta_0, E)$, which yields (3.36) for $j=1$. It remains to prove (3.35). To do so, we write the inequality (3.30) given by Proposition 3.6 for $(\mathbf{U}^{(1)}, h^{(1)})$, that is, for any $t \leqslant T_1$:

$$\|(\mathbf{U}^{(1)}, h^{(1)})(t)\|_2\,,\ \left(\int_0^t \|\mathbf{U}^{(j)}(\tau)\|_3^2\,\mathrm{d}\tau\right)^{1/2} \leqslant K\,e^{C\,(1+E)^2\,t}\left\{E/2 + \left(\int_0^t \|\mathbf{S}^{(0)}(\tau)\|_1^2\,\mathrm{d}\tau\right)^{1/2}\right\}.$$

Hence, applying Lemma 3.2 (3.19) to $\mathbf{S}^{(0)}$, we obtain

$$\|(\mathbf{U}^{(1)}, h^{(1)})(t)\|_2\,,\ \left(\int_0^t \|\mathbf{U}^{(j)}(\tau)\|_3^2\,\mathrm{d}\tau\right)^{1/2} \leqslant K\,e^{C\,(1+E)^2\,t}\left\{E/2 + C(\eta_0)\,E\,(1+E)\sqrt{t}\right\}.$$

Therefore, we can find $0 < T_2 = T_2(\eta_0, E) \leqslant T_1$ such that (3.35) is satisfied for any $t \leqslant T_2$. We choose $T := T_2$.

Next we pass from $j$ to $j+1$. If for any $j$ in $\mathbb{N}$, $(\mathbf{U}^{(j)}, h^{(j)})$ satisfies (3.34), (3.35) and (3.36) for any $t \leqslant T_2$, the existence of $(\mathbf{U}^{(j+1)}, h^{(j+1)})$ follows again from Proposition 3.6. Hence it remains to show (3.35) and (3.36) for $(\mathbf{U}^{(j+1)}, h^{(j+1)})$. As in the previous calculations, we rewrite the energy estimate (3.30) satisfied by $(\mathbf{U}^{(j+1)}, h^{(j+1)})$, that is for any $t \leqslant T$:

$$\|(\mathbf{U}^{(j+1)}, h^{(j+1)})(t)\|_2 \leqslant K\,e^{C\,(1+E)^2\,t}\left\{E/2 + \left(\int_0^t \|\mathbf{S}^{(j)}(\tau)\|_1^2\,\mathrm{d}\tau\right)^{1/2}\right\}.$$

Hence, applying again Lemma 3.2 (3.19) to $\mathbf{S}^{(j)}$ and using the bounds of $(\mathbf{U}^{(j)}, h^{(j)})$, it yields:

$$\|(\mathbf{U}^{(j+1)}, h^{(j+1)})(t)\|_2 \leqslant K\,e^{C\,(1+E)^2\,t}\left\{E/2 + C(\eta_0)\,E\left(1+E\right)\sqrt{t}\right\} \leqslant E,$$

since the same constants as previously are involved. In the same way we get (3.36) fo the rank $j+1$, this ends the proof, with $T = T_2$. $\square$

Now, to show that the sequence built above converges, we will prove that it is a Cauchy sequence in some function space. For this sake, for $j \geq 1$, we compute the difference between systems $(\mathcal{P}_{j+1})$ and $(\mathcal{P}_j)$:

$$(\mathcal{D}_j)\qquad \left\{\begin{array}{lcl} (\partial_t - 4\,\mu\,\partial_{xx})\left(\mathbf{U}^{(j+1)} - \mathbf{U}^{(j)}\right) & = & \mathbf{S}^{(j)} - \mathbf{S}^{(j-1)}\,, \\ \mathcal{L}_{u_N^{(j)}}\left(h^{(j+1)} - h^{(j)}\right) & = & F^{(j,j+1)} - F^{(j-1,j)} - \partial_x h^{(j)}\left(u_N^{(j)} - u_N^{(j-1)}\right), \\ \left(\mathbf{U}^{(j+1)} - \mathbf{U}^{(j)}, h^{(j+1)} - h^{(j)}\right)(t=0) & = & \mathbf{0}\,. \end{array}\right.$$

Let us rewrite the right hand side of the transport equation, denoted by $\tilde{F}$:

$$\begin{aligned} \tilde{F} &= F^{(j,j+1)} - F^{(j-1,j)} - \partial_x h^{(j)}\left(u_N^{(j)} - u_N^{(j-1)}\right) \\ &= -\sum_{k=1}^{N-1} \partial_x\left[h_i\left(u_i^{(j+1)} - u_i^{(j)}\right)\right] - h^{(j)}\,\partial_x\left(u_N^{(j+1)} - u_N^{(j)}\right) + \partial_x u_N^{(j)}\,(h^{(j)} - h^{(j-1)}) - \partial_x h^{(j)}\left(u_N^{(j)} - u_N^{(j-1)}\right). \end{aligned}$$

Thus, since the whole sequence is bounded by $E$, and by the use of the estimate (3.23) and Lemma 3.1 we get:

$$\|\tilde{F}\|_k \leqslant C\,E\,\left\|\left(\mathbf{U}^{(j)}-\mathbf{U}^{(j-1)},h^{(j)}-h^{(j-1)}\right)\right\|_k + C_b\,(1+E)\,\left\|\mathbf{U}^{(j)}-\mathbf{U}^{(j-1)}\right\|_{k+1},$$

for $k=1$ or 2. From Lemma 3.2 (3.22), we have also:

$$\left\|\mathbf{S}^{(j)}-\mathbf{S}^{(j-1)}\right\| \leqslant C(\eta_0)\,(1+E+E^2)\,\left\|\left(\mathbf{U}^{(j)}-\mathbf{U}^{(j-1)},h^{(j)}-h^{(j-1)}\right)\right\|_1.$$

Therefore, the solution to system $(\mathcal{D}_j)$ satisfies the following energy estimate, for any $t\leqslant T$ (where $T$ is given by Proposition 3.7):

$$\left\|\left(\mathbf{U}^{(j+1)}-\mathbf{U}^{(j)},h^{(j+1)}-h^{(j)}\right)(t)\right\|_1 \leqslant C(E)\,e^{C_b\,(1+E)^2\,t}\left(\int_0^t \left\|\left(\mathbf{U}^{(j)}-\mathbf{U}^{(j-1)},h^{(j)}-h^{(j-1)}\right)(\tau)\right\|_1^2\,\mathrm{d}\tau\right)^{1/2}.$$

Hence there exists a subsequence, still labelled $\left(\mathbf{U}^{(j)},h^{(j)}\right)_j$ such as

$$\left(\mathbf{U}^{(j)},h^{(j)}\right) \underset{j\to\infty}{\longrightarrow} (\mathbf{U},h) \text{ strongly in } \mathcal{C}(0,T;\mathbf{H}^1(\mathbb{R})).$$

Moreover, Lemma 3.7 gives, up to a subsequence, the convergence:

$$\mathbf{U}^{(j)} \underset{j\to\infty}{\rightharpoonup} \mathbf{U} \text{ weakly in } L^2\left(0,T;\mathbf{H}^3(\mathbb{R})\right),$$

while, for every fixed $t\leqslant T$:

$$\left(\mathbf{U}^{(j)},h^{(j)}\right)(t) \underset{j\to\infty}{\rightharpoonup} (\mathbf{U},h)(t) \text{ weakly in } \mathbf{H}^2(\mathbb{R}).$$

Thus we have a solution $(\mathbf{U},h)$ to system (3.18), lying in $\mathcal{C}(0,T;\mathbf{H}^1(\mathbb{R}))\cap L^\infty\left(0,T;\mathbf{H}^2(\mathbb{R})\right)$, satisfying for any $t,x$:

$$h(t,x) \geq \eta_0>0,$$
$$\|(\mathbf{U},h)(t)\|_2,\ \left(\int_0^t \|\mathbf{U}(\tau)\|_3^2\,\mathrm{d}\tau\right)^{1/2} \leqslant E.$$

Finally, we show that $(\mathbf{U},h)\in\mathcal{C}(0,T;\mathbf{H}^2(\mathbb{R}))$ by regularizing: we consider $(\mathbf{U}^\varepsilon,h^\varepsilon)=(\rho_\varepsilon*\mathbf{U},\rho_\varepsilon*h)$, where $\rho_\varepsilon*$ is the Friedrichs'mollifier with respect to $x$. Thus, applying $\rho_\varepsilon*$ to system (3.18) we obtain

$$\left\{\begin{aligned} &\partial_t\mathbf{U}^\varepsilon-\mu\,\partial_{xx}\mathbf{U}^\varepsilon=\mathbf{S}^\varepsilon+\mathbf{C}_0^\varepsilon,\\ &\partial_t h^\varepsilon+u_N^\varepsilon\,\partial_x h^\varepsilon=F^\varepsilon+C_1^\varepsilon,\\ &(\mathbf{U}^\varepsilon,h^\varepsilon)(t=0)=(\rho_\varepsilon*\mathbf{U}^0,\rho_\varepsilon*h^0)\in\mathcal{C}^\infty, \end{aligned}\right. \tag{3.37}$$

where $\mathbf{S}^\varepsilon=\rho_\varepsilon*\mathbf{S}$, $F^\varepsilon=\rho_\varepsilon*F$ and

$$\left\{\begin{aligned} \mathbf{C}_0^\varepsilon &= (\partial_t-\mu\,\partial_{xx})\,(\mathbf{U}^\varepsilon)-\rho_\varepsilon*(\partial_t-\mu\,\partial_{xx})\,(\mathbf{U}),\\ C_1^\varepsilon &= \{\partial_t h^\varepsilon-\partial_x\,(h^\varepsilon\,u_N^\varepsilon)\}-\rho_\varepsilon*\{\partial_t h-\partial_x\,(h\,u_N)\}. \end{aligned}\right.$$

By classical arguments on mollifiers [22, 28], we have, as $\varepsilon$ goes to zero:

$$\left\{\begin{aligned} &\mathbf{C}_0^\varepsilon,\ C_1^\varepsilon \ \to\ 0,\\ &(\mathbf{U}^\varepsilon,\,h^\varepsilon)\ \to\ (\mathbf{U},h). \end{aligned}\right.$$

Therefore, at the uniform limit we have $(\mathbf{U},h)\in\mathcal{C}(0,T;\mathbf{H}^2(\mathbb{R}))$. Uniqueness follows from the energy estimate and this concludes the proof of Theorem 1.1.

## 4. **Numerical scheme**

We propose in this section a discretization of the system (1.10) and present some numerical simulations. Due to the shallow water type formulation, we choose the Finite Volume framework. Many strategies are possible in this context (see [1, 3, 11] for approximate Riemann solvers; [2, 4] for kinetic schemes). Since we want to perform preliminary simulations to illustrate the dynamic behavior and the multilayer aspects of our model, we will use here a simple Finite Volume scheme, by isolating a "hyperbolic" part of the system, for which we can evaluate exact eigenvalues, without computing eigenvectors. The lawfulness of this choice can be discussed but the results obtained are good enough for the purpose we have.
In order to design our numerical scheme, we will consider a third formulation of the 1D multilayer problem (1.10). Now, the unknowns are denoted by $\mathbf{V}$, lying in $\mathbb{R}^{N+1}$, and $\mathbf{W}$ in $\mathbb{R}^N$:

$$= \left(V_i\right)_{0\leqslant i\leqslant N} = (h, h\,u_N, u_1, \ldots, u_{N-1})^T \ , \quad \mathbf{W} = (w_{1/2}, \ldots, w_{N-1/2})^T \, .$$

We separate the viscous terms. The horizontal one is included in the flux term with respect to $x$, and the vertical one is kept in the source term. It gives the formulation:

$$\begin{cases} \partial_t \mathbf{V} + \partial_x \mathbf{F}(\mathbf{V}) = \mathbf{S}(\mathbf{V}, \mathbf{W})\, , \\ w_{1/2} = u_1\, \partial_x z_b\, , \\ w_{i+1/2} - w_{i-1/2} = -h_i\, \partial_x u_i\, , \quad 1 \leqslant i \leqslant N-1\, . \end{cases} \tag{4.38}$$

The flux term $\mathbf{F} \in \mathbb{R}^{N+1}$ then comprises two parts: a convective part $\mathbf{F}^C$ corresponding to the transport and a diffusive one $\mathbf{F}^D$ corresponding to the horizontal viscosity. Precisely, we write its $i$th coordinate, for $0 \le i \le N$, as follows:

$$F_i \;=\; F_i^C + F_i^D\, ,$$

where

$$\begin{cases} F_0^C = h\, u_N\, , \\ F_1^C = h\, u_N^2 + g\, h^2/2\, , \\ F_i^C = u_{i-1}^2 + g\, h\, ,\ 2 \leqslant i \leqslant N\, . \end{cases} \qquad \begin{cases} F_0^D = 0\, , \\ F_1^D = -\mu\, h\, \partial_x u_N\, , \\ F_i^D = -\mu\, \partial_x u_{i-1}\, ,\ 2 \leqslant i \leqslant N\, . \end{cases}$$

The source term $\mathbf{S} = (S_i)_{0\leqslant i\leqslant N}$ is composed of three parts, coming from different effects, namely $\mathbf{G} = \mathbf{S}^b + \mathbf{S}^v + \mathbf{S}^e$. First the topography source term $\mathbf{S}^b$ is given by

$$\begin{cases} S_0^b = 0\, , \\ S_1^b = -g\, h\, \partial_x z_b\, , \\ S_2^b = \left( \dfrac{u_1^2}{h_1} - g \right)\, \partial_x z_b\, , \\ S_i^b = -g\, \partial_x z_b\, ,\ 3 \leqslant i \leqslant N\, . \end{cases}$$

Second, $\mathbf{S}^v$ represents the terms coming from the vertical viscosity and the friction:

$$\begin{cases} S_0^v = 0\,, \\[2mm] S_1^v = -2\,\mu\,\dfrac{u_N - u_{N-1}}{h + h_{N-1}}\,, \\[2mm] S_2^v = 2\,\mu\,\dfrac{u_2 - u_1}{h_1\,(h_1 + h_2)} - \dfrac{\kappa}{h_1}\,u_1\,, \\[2mm] S_i^v = 2\,\mu\,\dfrac{u_i - u_{i-1}}{h_{i-1}\,(h_i + h_{i-1})} - 2\,\mu\,\dfrac{u_{i-1} - u_{i-2}}{h_{i-1}\,(h_{i-1} + h_{i-2})}\,,\ 3 \leqslant i \leqslant N\,. \end{cases}$$

Finally $\mathbf{S}^e$ is the mass exchange term, given by:

$$\begin{cases} S_0^e = w_{N-1/2}\,, \\[2mm] S_1^e = u_{N-1/2}\,w_{N-1/2}\,, \\[2mm] S_2^e = -\dfrac{1}{h_1}\,u_{3/2}\,w_{3/2}\,, \\[2mm] S_i^e = \dfrac{1}{h_{i-1}}\,\left(u_{i-1/2}\,w_{i-1/2} - u_{i-3/2}\,w_{i-3/2}\right)\,,\ 3 \leqslant i \leqslant N\,. \end{cases}$$

The system is completed with the initial condition

$$\mathbf{V}(0,x) = \mathbf{V}^0(x) = \left(h^0, h^0\,u_N^0, u_1^0, \ldots, u_{N-1}^0\right)^T\,. \tag{4.39}$$

Then, we base the construction of the scheme on the following remark dealing with hyperbolicity of some part of the system (4.38).

**Remark 4.1.** *The system (4.38) when replacing the right hand side by* 0 *and taking* $\mu = 0$ *(that is only considering the transport flux* $\mathbf{F}^C$*), is hyperbolic and its eigenvalues read as:*

$$\mathcal{SP}(J) = \left\{u_N + \sqrt{g\,h}, u_N - \sqrt{g\,h}, 2u_1, \cdots, 2u_{N-1}\right\}\,.$$

It is simply seen when we compute the Jacobian $J$ of the flux:

$$\frac{\partial \mathbf{F}}{\partial \mathbf{V}} = J(\mathbf{V}) = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ gh - u_N^2 & 2u_N & 0 & 0 & \cdots & 0 \\ g & 0 & 2u_1 & 0 & \cdots & 0 \\ \vdots & 0 & 0 & 2u_2 & 0 & \cdots \\ \vdots & \vdots & \cdots & \cdots & \ddots & \cdots \\ g & 0 & \cdots & \cdots & \cdots & 2u_{N-1} \end{pmatrix}\,.$$

The eigenvalues are seen immediately. Moreover, the eigenvectors are computed easily, and the Jacobian matrix does not degenerate into a non diagonalizable matrix when some velocities become equal, as long as we have $h > 0$.

**Remark 4.2.** *This remark will be used to design the numerical scheme, but it is not really a property of hyperbolicity for the full system (1.10), since the source term of the formulation (4.38) contains derivatives of the unknowns* $u_i$*. Nevertheless the numerical results obtained with this formulation are totally lawful.*

We use now calligraphic letters to present the numerical scheme. We introduce a space-time discretization based on a uniform grid of points $x_{j+1/2}$ with space step $\Delta x$ and on a grid of points $t_k = k\,\Delta t$ with a time step $\Delta t$ wich will be precised later through a $CFL$ condition. The finite volume method consists in integrating the system on each control cell $C_j = (x_{j-1/2}; x_{j+1/2})$ of the mesh and each time step, and approximating the fluxes at the interfaces.

First, initial data $\left(\mathcal{V}_i^0\right)_{0\leqslant i\leqslant N}$ are given, for all $i \in \{0,\dots,N\}$, for all $j$ in $\mathbb{Z}$ by:

$$\mathcal{V}_{i,j}^0 = \frac{1}{\Delta x}\int_{C_j} V_i^0(x)\,\mathrm{d}x\,.$$

Then, the semi discrete numerical scheme reads, for all $i$ in $\{0,\dots,N\}$, for all $j$ in $\mathbb{Z}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{V}_{i,j} + \frac{1}{\Delta x}\left(\mathcal{F}_{i,\,j+1/2} - \mathcal{F}_{i,\,j-1/2}\right) = \mathcal{G}_{i,\,j}\,,$$

where $\mathcal{F}_{i,\,j+1/2}$ is the approximation of the $i$th coordinate of the flux at the cell interface $x_{j+1/2}$, while $\mathcal{G}_{i,\,j}$ and $\mathcal{V}_{i,\,j}$ are the approximations of the mean value of the $i$th coordinate of $\mathbf{G}$ and $\mathbf{V}$ on the cell $C_j$. Let us now describe the numerical flux, sum of the convective part and the diffusive part, first without slope limiters. On the one hand we choose a global Lax-Friedrichs method for the convective part, that is for all $i$ in $\{0,\dots,N\}$, for all $j$ in $\mathbb{Z}$:

$$\mathcal{F}_{i,\,j+1/2}^C = \frac{1}{2}\left[F_i^C(\mathcal{V}_{j+1}) + F_i^C(\mathcal{V}_j) - a_\infty\left(\mathcal{V}_{i,\,j+1} - \mathcal{V}_{i,\,j}\right)\right],$$

where $a_\infty = \sup\left\{|\lambda|, \lambda\in\mathcal{SP}(J)\right\}$. On the other hand, the diffusive flux, essentially an approximation of the gradient of the velocities at the cell interfaces, is discretized classically with a finite difference method: the coordinates of the discrete unknown $\mathcal{V}$ are expressed in terms of water height and velocities,

$$\mathcal{V}_0 = \mathcal{H}\,,\ \mathcal{V}_1 = \mathcal{H}\,\mathcal{U}_N\,,\ \mathcal{V}_i = \mathcal{U}_{i-1} \ \text{ for } \ 2\le i\le N\,,$$

so we define:

$$\begin{cases} \mathcal{F}_{0,\,j+1/2}^D = 0\,,\\[2ex] \mathcal{F}_{1,\,j+1/2}^D = -\mu\,\mathcal{H}_{j+1/2}\,\dfrac{\mathcal{U}_{N,\,j+1} - \mathcal{U}_{N,\,j}}{\Delta x}\,,\\[2ex] \mathcal{F}_{i,\,j+1/2}^D = -\mu\,\dfrac{\mathcal{U}_{i-1,\,j+1} - \mathcal{U}_{i-1,\,j}}{\Delta x}\,,\ 2\leqslant i\leqslant N\,,\end{cases}$$

where $\mathcal{H}_{j+1/2}$ is an approximation of function $h$ at the cell interface $x_{j+1/2}$. We choose the harmonic mean of $\mathcal{H}_j$ and $\mathcal{H}_{j+1}$ [14], that is:

$$\mathcal{H}_{j+1/2} = \frac{2\,\mathcal{H}_j\,\mathcal{H}_{j+1}}{\mathcal{H}_{j+1} + \mathcal{H}_j}\,.$$

Before treating the source terms, we shall impose the $CFL$ stability condition:

$$a_\infty\,\frac{\Delta t}{\Delta x} + 2\,\mu\,\frac{\Delta t}{\Delta x^2} \,<\, 1\,. \tag{CFL}$$

This choice can be somehow justified by the next calculations. To simplify, let us take the source term equal to zero and consider an explicit Euler scheme in time. Then, with the choice of three points explicit fluxes we made, we may write the numerical solution at time $t^{n+1}$, $\mathcal{V}_{i,j}^{n+1}$ as a combination of $\mathcal{V}_{i,j+1}^n$, $\mathcal{V}_{i,j}^n$ and $\mathcal{V}_{i,j-1}^n$. Precisely, we have, for $j\in\mathbb{Z}$, for $0\le i\le N$:

$$\mathcal{V}_{i,j}^{n+1} \;=\; \alpha_{i,j}^n\,\mathcal{V}_{i,j-1}^n + \beta_{i,j}^n\,\mathcal{V}_{i,j}^n + \gamma_{i,j}^n\,\mathcal{V}_{i,j+1}^n\,,$$

where

$$\begin{cases} \alpha_{i,j}^n = \partial_i F_i^C(\xi_i^n)\,\dfrac{\Delta t}{2\,\Delta x} + a_\infty\,\dfrac{\Delta t}{2\,\Delta x} + \mu\,\dfrac{\Delta t}{\Delta x^2}\,\delta_{i,j-1/2}^n\,,\\[2ex] \beta_{i,j}^n = 1 - a_\infty\,\dfrac{\Delta t}{\Delta x} - \mu\,\dfrac{\Delta t}{\Delta x^2}\,\left(\delta_{i,j-1/2}^n + \delta_{i,j+1/2}^n\right)\,,\\[2ex] \gamma_{i,j}^n = -\partial_i F_i^C(\xi_i^n)\,\dfrac{\Delta t}{2\,\Delta x} + a_\infty\,\dfrac{\Delta t}{2\,\Delta x} + \mu\,\dfrac{\Delta t}{\Delta x^2}\,\delta_{i,j+1/2}^n\,,\end{cases}$$

$$\text{with } \gamma^n_{i,j+1/2} = \begin{cases} 0 & \text{if } \; i = 0\,, \\ \mathcal{H}^n_{j+1/2} & \text{if } \; i = 1\,, \\ 1 & \text{if } \; 2 \le i \le N\,. \end{cases}$$

Now, with the definition of $a_\infty$ and the fact that the size of the highest layer is of order $O\left(\overline{h}\right)$ strictly smaller than 1, the condition (CFL) makes sens. Instead of an Euler scheme, we will rather use an explicit Runge Kutta of order 4 scheme in time. Moreover, we introduce slope limiters in the fluxes to reconstruct the unknowns $\mathcal{V}_i$ on the right (+) and on the left (−) of the cell interfaces $x_{j+1/2}$, in order to reduce the numerical diffusion inherent in Lax-Friedrichs schemes. We use the classical minmod limiters $\sigma_{i,j}$, that is for $0 \le i \le N$, for $j \in \mathbb{Z}$:

$$\sigma_{i,j} = \text{ minmod } \left(\mathcal{V}_{i,j+1} - \mathcal{V}_{i,j}, \mathcal{V}_{i,j} - \mathcal{V}_{i,j-1}\right).$$

Then the right and left reconstructed values at interface $x_{j+1/2}$ of the $i$th coordinate of $\mathcal{V}$ read:

$$\begin{cases} \mathcal{V}^+_{i,j+1/2} = \mathcal{V}_{i,j+1} - \sigma_{i,j+1} \left(\mathcal{V}_{i,j+2} - \mathcal{V}_{i,j+1}\right)\,, \\[2ex] \mathcal{V}^-_{i,j+1/2} = \mathcal{V}_{i,j} + \sigma_{i,j} \left(\mathcal{V}_{i,j+1} - \mathcal{V}_{i,j}\right)\,. \end{cases}$$

Hence the convective flux including the slope limiters reads, for $0 \le i \le N$, for $j \in \mathbb{Z}$:

$$\mathcal{F}^C_{i,j+1/2} = \frac{1}{2}\left[F^C_i(\mathcal{V}^+_{j+1/2}) + F^C_i(\mathcal{V}^-_{j+1/2}) - a_\infty \left(\mathcal{V}^+_{i,j+1/2} - \mathcal{V}^-_{i,j+1/2}\right)\right].$$

We do the same to compute the diffusive flux $\mathcal{F}^D$. Let us now define the discrete $\mathcal{W}$ variable. Since the vertical velocity $\mathcal{W}_{i+1/2,j}$ is essentially an horizontal gradient of the velocity at the layer interface $z_{i+1/2}$ and in the horizontal cell $C_j$, we choose the following reconstruction for $j \in \mathbb{Z}$, for $1 \leqslant i \leqslant N-1$:

$$\begin{cases} \mathcal{W}_{1/2,j} = \mathcal{V}_{3,j}\, \partial_x z_b(x_j)\,, \\[2ex] \mathcal{W}_{i+1/2,j} \; = \mathcal{W}_{i-1/2,j} - h_i\, \dfrac{1}{2\,\Delta x} \left\{\left(\mathcal{U}^+_{i,j+1/2} + \mathcal{U}^-_{i,j+1/2}\right) - \left(\mathcal{U}^+_{i,j-1/2} + \mathcal{U}^-_{i,j-1/2}\right)\right\}. \end{cases}$$

Finally we choose a smooth topographic source term, for which we can compute the derivative exactly. This avoids numerical difficulties that can occur with the discretization of the source term $\mathbf{S}^b$ (which can be handled by different methods, such as the hydrostatic reconstruction [3, 16]).

**Remark 4.3.** *We can mention that an explicit scheme in time may be very restrictive because of the viscous terms. Nevertheless, we consider a small viscosity coefficient, which reduces the constraint on the time step. We can also note that this numerical scheme preserves at the discret level the conservation of the mass* $\int h\, dx$ *and the constant steady states with periodic boundary condition, as long as the initial number of layers is conserved.*

## 5. **Numerical experiments**

In this section we present a few numerical simulations performed to validate numerically the multilayer system and its discretization. We also show its dynamic behavior, depending on the small parameter $\overline{h}$ chosen initially to discretize in the vertical direction. In all the tests performed, the $CFL$ number is equal to 0.95. In the first test we simply verify the consistency of our model with the classical shallow water system under shallow water assumption. The second test aims at comparing the multilayer aspects when we choose different values of the size $\overline{h}$ of the inside layers. In the last 3 tests, we propose to observe the dynamic behavior of the model, adding and removing layers to adapt to different kinds of flows over different topographies.

5.1. **Test 1: perturbation of rest in velocity, flat bottom.** In this section, we consider a flat bottom, the spatial domain is $[0, 1]$, viscosity $\mu = 0.0001$, no friction, periodic boundary conditions and we perturb the lake at rest in velocity, taking for initial conditions:

$$\begin{cases} \eta(0,x) \equiv 0.1\,, \quad u_i(0,x) \equiv 0 \;\text{ for }\; 1 \leqslant i \leqslant N-1\,, \\ \\ u_N(0,x) = 0.2\,\sin(2\,\pi\,x)\,. \end{cases} \tag{5.40}$$

Then we run the multilayer code for different numbers of layers: 5 (amplitude $\overline{h} = 0.02$) and 15 layers ($\overline{h} = 0.006$). Thus, in Figure 3, we show the evolution in time of the free surface and the velocity field inside the fluid, for these two choices. We observe the multilayer aspects of the velocity field, that is appearance of vorteces, becoming more visible when we take a larger number of layers.

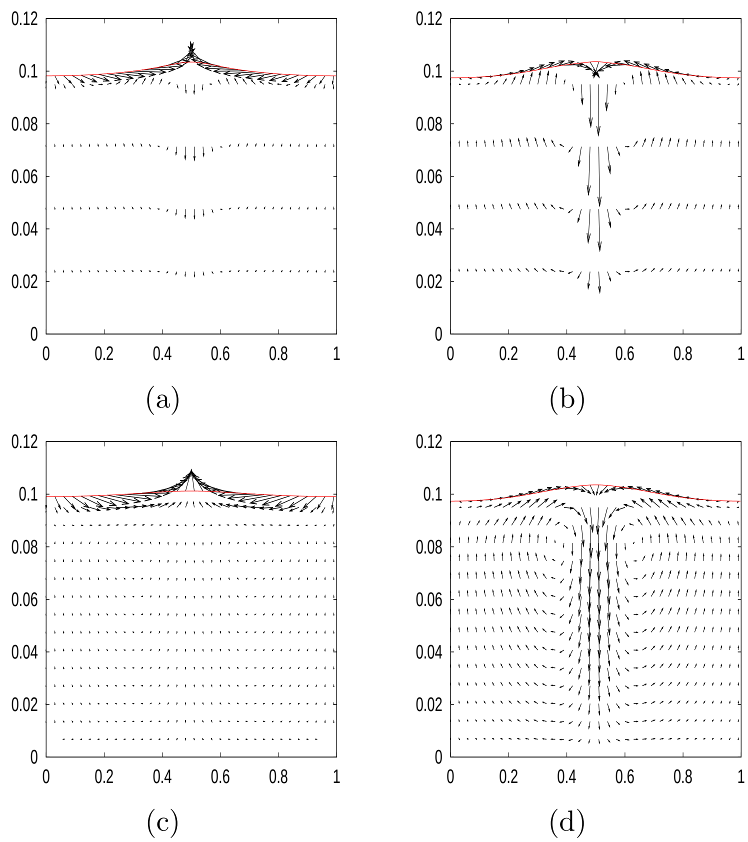


FIGURE 3. *Evolution of the velocity field for initial data (5.40)for $t = 1$ (left) and $t = 10$ (right): 5 layers (a),(b), and 15 layers (c), (d).*

5.2. **Test 2: dynamic behavior, non flat bottom.** For this test case we consider a spatial domain $[-5, 5]$, viscosity $\mu = 0.00005$, friction $\kappa = 0.0001$, size of the inside layers $\overline{h} = 0.07$ and a smooth rapidly oscilatting bottom

$$z_b(x) = 0.09\,\sin(6\,\pi\,x/20)\,.$$

We perturb first the rest with an initial constant velocity:

$$\begin{cases} \eta(0,x) = 1.2\,, \\ \\ u_i(0) \equiv 0.2 \;\text{ for }\; 1 \leqslant i \leqslant N\,. \end{cases} \tag{5.41}$$

We then output in Figure 4 the velocity field and the free surface at different times. We have initially 15 layers with this choice, then the free surface starts to oscillate and the inside fluid starts to circulate:

the number of layers then varies, (decreases and increases), before stabilizing in a stable state where the free surface mimics the shape of the topography, around $t = 50$.

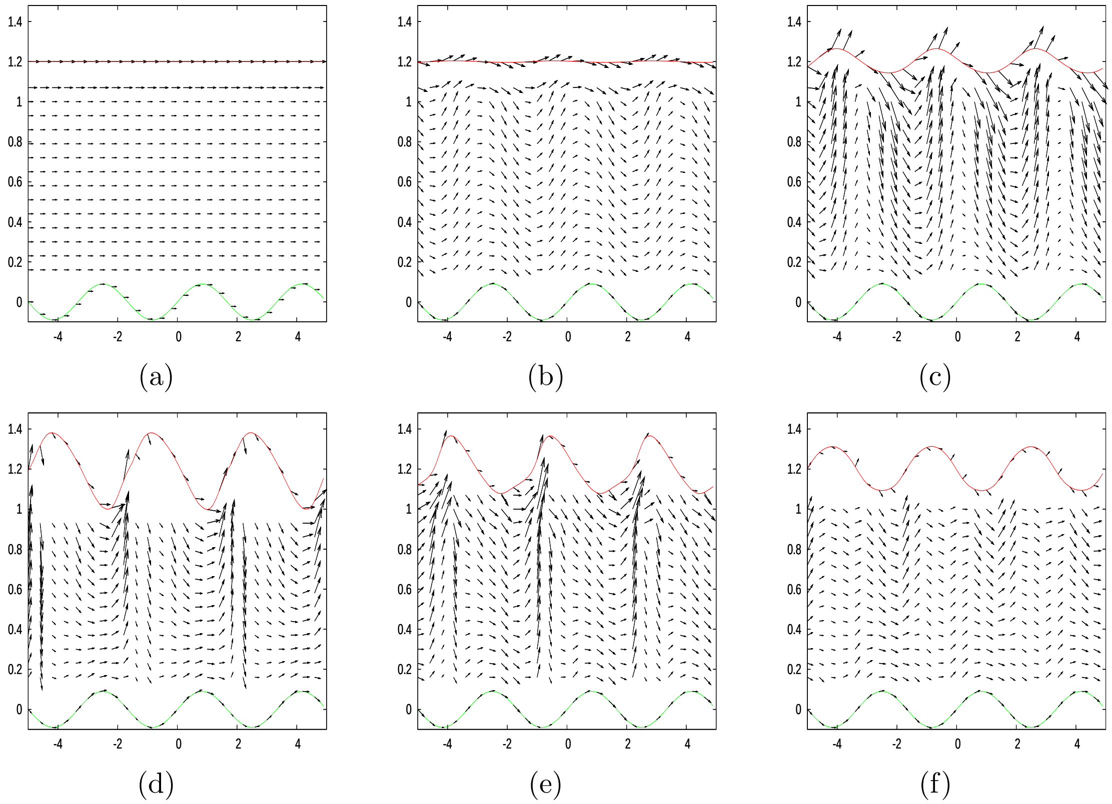


Figure 4. *Free surface and velocity field for initial data (5.41) at times $t = 0$ (15 layers)(a), $t = 0.1$ (15 layers) (b), $t = 0.5$ (15 layers)(c),$t = 1.5$ (13 layers)(d), $t = 5$ (15 layers) (e) and final time $t = 50$ (14 layers)(f).*

## References

[1] Audusse, E., A multilayer Saint-Venant model: derivation and numerical validation, Discrete Contin. Dyn. Syst. Ser. B (2005)

[2] Audusse, E. and Bristeau, M. O. and Decoene, A., Numerical simulations of 3D free surface flows by a multilayer Saint-Venant model, Internat. J. Numer. Methods Fluids (2008)

[3] Audusse, E. and Bristeau, M.-O., Finite-volume solvers for a multilayer Saint-Venant system, Int. J. Appl. Math. Comput. Sci. (2007)

[4] Audusse, E., Bristeau, M. O., Perthame, B. and Sainte-Marie, J., A multilayer Saint-Venant system with mass exchanges for Shallow Water flows: derivation and numerical validation, M2AN Math. Model. Numer. Anal. (2010)

[5] Azèrad, P. and Guillén F., Mathematical justification of the hydrostatic approximation in geophysical fluid dynamics, SIAM J. Math. Anal. (2001)

[6] Bouchut F. and Westdickenberg M., Gravity driven shallow water models for arbitrary topography, Comm. in Math. Sci. (2004)

[7] Boutounet M., Chupin L., Noble P. and Vila, J.P., Shallow water flows for arbitrary topography, Comm. Math. Sciences, 6 (2008)

[8] Bresch, D., Guillén-González, F., Masmoudi, N. and Rodríguez-Bellido, M.A., Asymptotic derivation of a Navier condition for the primitive equations, Asympt. Anal.

[9] Bresch, D. and Desjardins, B., Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model, Comm. Math. Phys. (2003)

[10] Bui, A. T., Existence and uniqueness of a classical solution of an initial boundary value problem of the theory of shallow waters, SIAM J. Math. Anal. (1981)

[11] Castro, M., Macías, J. and Parés, C., A $Q$-scheme for a class of systems of coupled conservation laws with source term. Application to a two-layer 1-D shallow water system, M2AN Math. Model. Numer. Anal.(2001)
[12] Castro, M. J. and García-Rodríguez, J. A. and González-Vida, J. M. and Parés, C., A parallel 2D finite volume scheme for solving the bilayer shallow-water system: modellization of water exchange at the Strait of Gibraltar, Parallel computational fluid dynamics, Elsevier B. V., Amsterdam (2005)
[13] Decoene, A., Bonaventura, L., Miglio, E. and Saleri, F., Asymptotic derivation of the section-averaged shallow water equations for river hydraulics, MOX-Report (2007)
[14] Eymard R., Gallouët, T. and Herbin R., Finite volume methods, Handbook of numerical analysis (2000)
[15] Ferrari, S. and Saleri, F., A new two-dimensional Shallow Water model including pressure effects and slow varying bottom topography, M2AN Math. Model. Numer. Anal. (2004)
[16] Gallouët, T., Hérard J-M. and Seguin, N., Some approximate Godunov schemes to compute shallow-water equations with topography, Computers and Fluids (2003)
[17] Gerbeau, J.-F. and Perthame, B., Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation, Discrete Contin. Dyn. Syst. Ser. B (2001)
[18] Kloeden, P.E., Global existence of classical solutions in the dissipative shallow water equations, SIAM J.Math. Anal. (1985)
[19] Lions, P.L., Mathematical Topics in Fluid Mechanics, (1) Oxford University Press (1996)
[20] Lions, P.L., Temam, R. and Wang S., On the equations of the large-scale ocean Nonlinearity (1992)
[21] Marche, F., Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects, European Journal of Mechanic (2007)
[22] Matsumura, A. and Nishida, T., The initial value problem for the equations of motion of viscous and heat-conductive gases, Journal of Mathematics of Kyoto University (1980)
[23] Pedlosky, J., Geophysical fluid dynamics, Springer (1987)
[24] Perthame, B. and Simeoni, C., A kinetic scheme for the Saint-Venant system with a source term, Calcolo, 38 (2001)
[25] Stoker J.J., Water waves, the mathematical theory with applications, Wiley (1958)
[26] Sundbye, L., Global existence for Dirichlet problem for the viscous shallow water equations, J. Math. Anal. Appl. (1996)
[27] Sundbye, L., Global existence for the Cauchy problem for the viscous shallow water equations, Rocky Mt. J. Math. (1998)
[28] Taylor, Michael E., Partial differential equations. III, Springer-Verlag (1997)
[29] Wang, W. and Xu, C., The Cauchy problem for viscous shallow water equations, Rev. Mat. Iberoamericana (2005)
[30] Whitham G. B., Linear and nonlinear waves, John Wiley and Sons Inc., New York (1999)